Physics 471 – Introduction to Modern Atomic Physics
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HOMEWORK III (Due: 11/22/2006)

(Please answer 6 of the following 10 questions.)

1. Bose-Einstein condensation of ideal bosons

N weakly interacting bosons $H = \frac{p^2}{2m}$ are confined in a square box with size L.

A. At temperature $T > T_c$, what is the population in the ground state N_0?

B. What is the critical temperature (s.t. $N_0=1$) and the phase space density $n\lambda_{dB}$ at $T=T_c$? (n is density, λ_{dB} is thermal deBroglie wavelength.)

C. Now in a harmonic trap $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$. What is the critical temperature?

D. What is $n\lambda_{dB}$ at the center of the trap when $N_0=1$?

E. What is the phase density of the oxygen we are breathing?

2. BEC in a box – healing length

Consider a BEC in a large box with mass m, mean density n and scattering length a.

A. What is the density distribution when a=0?

B. Near the box, show that the density goes like $\sim \tanh^2(r/\sqrt{2}\xi)$, where r is the distance from the wall and $\xi = (8\pi na)^{-1/2}$ is the healing length.

(Hint: Solve G-P equation with boundary conditions $\psi(r=0)=0$ and $\psi(r=\infty)=const.$)

C. How do you deform a box s.t. the ground state wavefunction of a particle approaches a constant?

3. Can we condense bosonic ions?

First, consider 2 bosonic ions with charge +e in a 1D box of size L at T=0.

A. What is the ground state of the system?

B. What is the ground state energy of He if the two electrons are bosons?

C. Comments on whether bosonic ions in the continuum with density n can condense?

4. Free expansion of BEC

Consider a BEC described by the G-P equation $i\hbar\partial_t\psi = (V + g|\psi|^2)\psi$ (with Thomas-Fermi approximation), chemical potential μ, $V = \frac{1}{2}m(\omega_r^2 r^2 + \omega_z^2 z^2)$. At t=0, the potential is suddenly switched off.

A. What is the density distribution $\rho(r,t)$ at $t=0^+$ and the acceleration of atoms at (r, z)?

(Hint: In free space, force is determined by the gradient of the mean-field energy.)

B. Use Newton's law and show that this force leads to merely a dilatation of the wavefunction, namely, $\rho(r,z,t) = R(t)^{-2} Z(t)^{-1} \rho(r/R(t), z/Z(t), 0)$.

C. Show that $\ddot{R} = \frac{\omega_r^2}{R^3 Z}$ and $\ddot{Z} = \frac{\omega_z^2}{R^2 Z^2}$.

D. Solve *R(t)* and *Z(t)* with (i) $\omega_r = \omega_z$ and(ii) $\omega_r >> \omega_z = \varepsilon\omega_r > 0$

5.Collapse of BEC with attractive interaction

Consider a BEC in a spherical trap $V = \frac{1}{2} m\omega_r^2 r^2$ with an attractive interaction $a<0$.

A. Estimate the critical atom number which can lead to the collapse of the condensate.
B. Assume a=0, can gravity lead to collapse of the BEC? If yes, estimate the critical number of atoms?

6. Square lattice and triangular lattice

A. Write a simple program (Mathematica) to show that you can create a square lattice with 4 laser beams on the x-y plan. Comment on the polarization (E-field direction) of your choice?
B. Show that you can create a triangular lattice with 3 beams on the x-y plane.

7. Atoms in optical lattices

Given a 1D lattice potential $V = V_0 \sin^2 kx$, $V_0>0$

A. What is the ground state energy in the tight-binding regime?
B. What is the onsite energy when two atoms are in the ground state?
(Hint: treat interaction perturbatively.)
C. What is the onsite energy when N atoms are in the ground state?

8. Bose-Hubbard model.

Bosons in an lattice can be described by (b_i^+ is the creation operation of a boson at site *i*)

$$H = -t \sum_{\langle i,j \rangle} \left(b_i^\dagger b_j + b_j^\dagger b_i \right) + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i .$$

Determine t, U and μ for atoms in a 3D lattice $V = V_0 (\sin^2 kx + \sin^2 ky + \sin^2 kz)$.

(Hint: Consider T=0, $V = V_0 \sin^2 kx$ in the tight binding regime.)

9. "Two-particle" superfluid to Mott insulator transition in an optical lattice

The Hamiltonian in 8 can describe two atoms in a double well potential (|L> & |R>).
"Superfluid"state is $|S> = \frac{1}{2}(b_L^+ + b_R^+)^2$; "Mott insulator" state is $|M> = b_L^+ b_R^+$.

A. Write the two states in the basis of |L,L>, |L,R>, |R,L> and |R,R>.
B. Which state has a lower energy, does that depend on t/U?
C. What is the critical t/U when the two states have the same energy?
D. Generalize this calculation to 3 and many particles and determine the critical t/U?

10. Momentum distribution of a superfluid in an optical lattice

Given a superfluid in a 1D lattice potential $V = V_0 \sin^2 kx$. When the interaction is weak, the system remains superfluid in the tight-binding regime.

A. How would you write down the superfluid wavefunction in this regime?
B. What is the wavefunction in momentum space?
C. How does the momentum wavefunction change when V_0 increases?