

HOMEWORK I

(Due: 10/20/2006)

(Please answer 6 of the following 10 questions.)

1. Quantum states in a 3D Gaussian potential

Consider $H = \frac{p^2}{2m} - A \exp(-2r^2 / r_0^2)$, where A is a constant.

- How large should A be to support a bound state?
- When A is much larger than that value, estimate how many bound states are there?
- Following B., what are the densities of states at the center and near the rim?

2. Creation of coherent state and squeezed state

An oscillator, described by $H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2$ is initially in the ground state.

- At $t = 0$, we instantaneously displace the trap by a , show that the particle is transferred to a coherent state $|b\rangle$ in the new trap.

(Hint: show $\hat{a}|b\rangle = b|b\rangle$ and determine b .)

- Determine $|b(t)\rangle$.

3. Phase-shift due to Landau-Zener probe (Difficult)

Consider $H = \begin{bmatrix} Pt & \hbar\Omega \\ \hbar\Omega & 0 \end{bmatrix}$, where $P = \text{const.}$ and t is time. At $t = -\infty$, the system is in the zero energy state $|t = -\infty\rangle = |E = 0\rangle$. Determine $|t = +\infty\rangle$ to leading order in $\Omega \ll \sqrt{P/\hbar}$.

4. Ramsey spectroscopy

Derive the Ramsey lineshape showed on page 19 of Lecture 1 based on the scheme shown on page 14 of Lecture 2.

5. Dark state

Construct a dressed state basis $|A; N_1, N_2\rangle$ for the three-level system on page 17 of Lecture 2, where $A = a, b$ or c is the quantum number of the atoms, N_1 and N_2 are the photon numbers of the two radiation modes.

- What are the dark states in the following cases: 1. $\Omega_a \neq 0$ and $\Omega_b = 0$, 2.

$\Omega_a = 0$ and $\Omega_b \neq 0$ and 3. $\Omega_a \neq 0$ and $\Omega_b = 0$.

- In the third case, what are the other two “bright” eigen-states?

6. Atomic structure

- Why is periodic table filled in the order of 1s 2s 2p 3s 3p 4s 3d 4p 5s...?
- Use mathematica to plot out one of the angular wave functions shown on page 4.

c. Consider an $nsnp$ configuration, sketch the energy scheme from LS coupling to jj coupling. (Hint: Compare page 6 and page 13 of Lecture 3).

7. Zeeman shifts in the ${}^6\text{Li}$ ground state

Derive the eigen-energies in the ground state hyperfine manifold of ${}^6\text{Li}$ in the presence of a magnetic field B . (Hint: See Page 11 of Lecture 3 or you can check <http://www.phy.duke.edu/research/photon/qoptics/techdocs/pdf/PropertiesOfLi.pdf>.)

8. Fine structure of atoms

Consider the $2p$ state of a hydrogen atom in the absence of external fields. Spin-orbit interaction is given by $H_{s-o} = A \mathbf{s} \cdot \mathbf{L}$

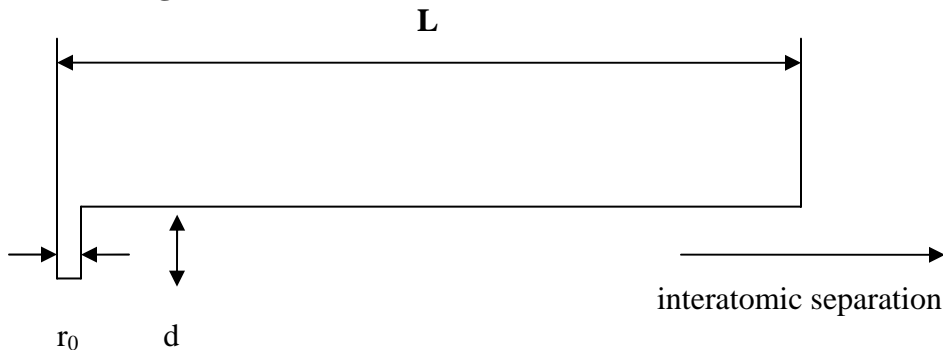
- Express the eigen-energies and eigen-states in terms of $|s = 1/2, L = 1; m_s, m_L \rangle$
- If you wish to radiatively couple $|1s_{1/2}, m_j = 1/2 \rangle$ to $|2p_{3/2}; m_j = 3/2 \rangle$, how would you orient the electric field (polarization) of the laser beam (Hint: Choose Z axis as your quantization axis.)
- What is the ratio of the following dipole transition strengths: $|1s_{1/2}, m_j = 1/2 \rangle \rightarrow |2p_{3/2}; m_j = 1/2 \rangle$ and $|1s_{1/2}, m_j = 1/2 \rangle \rightarrow |2p_{1/2}; m_j = 1/2 \rangle$?
- Now consider a carbon atom in the configuration of $2p^2$. Determine the terms and eigen-states of the fine structure in the LS-coupling scheme.

9. Thermal gas in a harmonic potential

Consider a harmonically trapped gas, $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$, at temperature T .

- What are the distributions of density and momentum of the gas.
- Repeat A in a linear potential $H = \frac{p^2}{2m} + Ar$.

10. Scattering in a box



Approximate the continuum by a large box with size L and the interaction range is $r_0 \ll L$.

- What is the critical $d = d_c$ when the 1st bound state forms in the interaction potential?
- Sketch the ground state wave function of the system at $d = 0.99d_c$.
- Sketch the bound state and the ground state in the "continuum box" at $d = 1.01d_c$.
- Take $L/r_0 \rightarrow \infty$, determine scattering length a at $d = 0.99d_c$, $1.00d_c$, and $1.01d_c$.

1. Quantum states in a 3D Gaussian potential

A.

To find the first bound state, we consider only s-wave scattering. The Schrodinger equation is

$$u'' + \frac{2m}{\hbar^2} (A \exp(-2r^2 / r_0^2) + E)u = 0$$

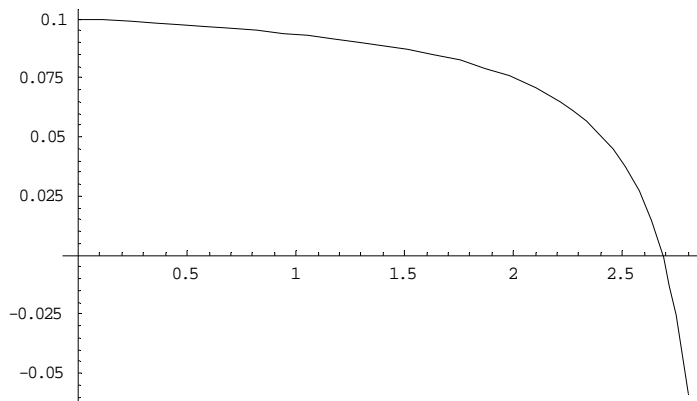
Or can be written as

$$\frac{d}{dr} (u' / u) + (u' / u)^2 = -\frac{2m}{\hbar^2} (A \exp(-2r^2 / r_0^2) + E)$$

We take the limit of $E \rightarrow 0$ and evaluate u'/u numerically at sufficiently large r (say few r_0). For a zero energy bound state, the wave function decays to zero at infinity. Thus,

u'/u will tends to zero for sufficiently large r . The figure below is $r_0 \frac{u'(10r_0)}{u(10r_0)}$ v.s.

$\frac{2mr_0^2}{\hbar^2} A$, and the first bound state appears at $\frac{2mr_0^2}{\hbar^2} A \approx 2.68396$



From the definition of scattering length, a , $\frac{u'(r)}{u(r)} \xrightarrow{E \rightarrow 0} \frac{1}{r - a}$, where r is the range of the

scattering potential(In our case, since our potential is a Gaussian and extends to infinity, we take some large $r = 10 r_0$). When a bound state appears, the scattering length diverges and, thus, u'/u vanishes.

We can also use WKB method to calculate the energy level for comparison. (Although

for the first bound state, it is a poor approximation since $\frac{d}{dr} \frac{\hbar}{\sqrt{2mA \exp(-2r^2 / r_0^2)}} \approx 1$)

For first bound state to appear, we need the condition

$$\int_0^\infty \sqrt{\frac{2m}{\hbar^2} A \exp(-2r^2 / r_0^2)} dr = 1 / 2\pi$$

And this gives us $A = \frac{\pi\hbar^2}{2mr_0^2}$, a bit larger than the numerical solution.

B.

If A is much larger then the value we obtained above, we could estimate the number of bound states semiclassically. From uncertainty principle, $\Delta x \Delta p \sim \hbar$, each bound state occupies \hbar^3 volume in the phase space. The number of bound states N is

$$\begin{aligned} N &\approx \frac{1}{\hbar^3} \int dx dy dz dp_x dp_y dp_z \\ &= \frac{(4\pi)^2}{3\hbar^3} \int p^3 r^2 dr \\ &= \frac{(4\pi)^2}{3\hbar^3} (2mA)^{3/2} \int \exp(-3r^2/r_0^2) r^2 dr \\ &= \frac{4\pi}{3} r_0^3 \left(\frac{2mA\pi}{3\hbar^2}\right)^{3/2} \end{aligned}$$

C.

In B, we replace p^2 by $-2mV(r)$ to integrate the phase space volume occupied by states with energy lower than $E=0$ and we obtain the total bound state number N. If we replace p^2 by $2m(E-V(r))$, we obtain the total number $N(E)$ for states with energy below E. That is

$$N(E) \approx \frac{(4\pi)^2}{3\hbar^3} (2mA)^{3/2} \int_0^{r_E} \left(e^{-2r^2/r_0^2} + \frac{E}{A}\right)^{3/2} r^2 dr$$

Where,

$$E = V(r_E) = -A \exp(-2r_E^2/r_0^2),$$

$$\text{Since } N(E) = \int^E \rho(E) dE, \quad \rho(E) = \frac{dN(E)}{dE}$$

Density of states

$$\begin{aligned} \rho(E) &\approx \frac{(4\pi)^2}{3\hbar^3} (2mA)^{3/2} \frac{\partial}{\partial E} \int_0^{r_E} \left(e^{-2r^2/r_0^2} + \frac{E}{A}\right)^{3/2} r^2 dr \\ &= \frac{(4\pi)^2}{3A\hbar^3} (2mA)^{3/2} \left[\frac{3}{2} \int_0^{r_E} \left(e^{-2r^2/r_0^2} + \frac{E}{A}\right)^{1/2} r^2 dr + \frac{\partial r_E}{\partial E} \overbrace{\left(e^{-2r_E^2/r_0^2} + \frac{E}{A}\right)^{1/2} r_E^2} \right] \\ &= \frac{8\pi^2}{A\hbar^3} (2mA)^{3/2} \int_0^{r_E} \left(e^{-2r^2/r_0^2} + \frac{E}{A}\right)^{1/2} r^2 dr \end{aligned}$$

Thus, at the center of the trap, $r_E = 0$, $\rho(E) = 0$

Near the rim of the trap, $E \rightarrow 0$, $\rho(E) \approx \frac{2\pi r_0^3}{A\hbar^3} (2m\pi A)^{3/2}$

2. Creation of coherent state and squeezed state

A.

We can say that instantaneously displacing the trap by a is equivalent to shifting the state by $-a$. This can be done by applying the translation operator $e^{i\hat{p}a/\hbar}$ to the ground state $|0\rangle$, where \hat{p} is the momentum operator and could be written in terms of the creation and annihilation operators of the same trap:

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger)$$

After shifting, the original ground state becomes

$$|b\rangle = e^{i\hat{p}a/\hbar} |0\rangle = \exp\left(\sqrt{\frac{m\omega}{2\hbar}}a(\hat{a} - \hat{a}^\dagger)\right) |0\rangle$$

Where

$$\exp\left(\sqrt{\frac{m\omega}{2\hbar}}a(\hat{a} - \hat{a}^\dagger)\right) = e^{-\sqrt{\frac{m\omega}{2\hbar}}a\hat{a}^\dagger} e^{\sqrt{\frac{m\omega}{2\hbar}}a\hat{a}} e^{1/2\frac{m\omega}{\hbar}a^2[\hat{a}^\dagger, \hat{a}]}$$

Thus the transferred state is $|b\rangle = e^{-1/2\frac{m\omega}{\hbar}a^2} e^{-\sqrt{\frac{m\omega}{2\hbar}}a\hat{a}^\dagger} |0\rangle$. Here, we have used

$[\hat{a}, \hat{a}^\dagger] = 1$ and $\hat{a}|0\rangle = 0$. Expand $|b\rangle$ into the energy eigenstates of the trap, we get

$|b\rangle = e^{-1/2b^2} \sum_{n=0}^{\infty} \frac{b^n}{\sqrt{n!}} |n\rangle$. This is by definition a coherent state. It can easily be seen

that $\hat{a}^\dagger |b\rangle = b |b\rangle$ with $b = -\sqrt{\frac{m\omega}{2\hbar}}a$

B.

The time evolution of this state is just the summation of the time evolution of all the eigenstates.

$$\begin{aligned} |b(t)\rangle &= e^{-1/2b^2} \sum_{n=0}^{\infty} \frac{b^n}{\sqrt{n!}} e^{i(n+1/2)\omega t} |n\rangle \\ &= e^{-1/2b^2 + i\omega t/2} \sum_{n=0}^{\infty} \frac{(be^{i\omega t})^n}{\sqrt{n!}} |n\rangle \\ &= e^{i\omega t/2} |be^{i\omega t}\rangle \end{aligned}$$

3

For the following few problems we use

$$\hbar = 1 \quad (1)$$

Let the state vector be

$$\begin{pmatrix} a_1(t) \\ b_1(t) \end{pmatrix}$$

The Schrödinger's equation is

$$\begin{aligned} i\dot{a}_1 &= Pta_1 + \Omega b_1 \\ i\dot{b}_1 &= \Omega b_1 \end{aligned} \quad (2)$$

where $\Omega \ll 1$.

Introducing the substitution

$$\begin{aligned} a_1 &= a_2 e^{-iPt^2/2} \\ b_1 &= b_2 \end{aligned} \quad (3)$$

with the initial condition

$$\begin{aligned} a_2(t = -\infty) &= 0 \\ b_2(t = -\infty) &= 1 \end{aligned} \quad (4)$$

we get

$$\begin{aligned} \dot{a}_2 &= -ie^{iPt^2/2} \Omega b_2 \\ \dot{b}_2 &= -i\Omega a_2 e^{-iPt^2/2} \end{aligned} \quad (5)$$

Introducing the time scale change $\sqrt{P}t = s$, we get

$$\begin{aligned} \frac{da_2}{ds} &= -ie^{is^2/2} \frac{\Omega}{\sqrt{P}} b_2 \\ \frac{db_2}{ds} &= -ie^{-is^2/2} \frac{\Omega}{\sqrt{P}} a_2 \end{aligned} \quad (6)$$

The 1st order time-dependent perturbation gives

$$\begin{aligned} a_2(+\infty) &= -i \frac{\Omega}{\sqrt{P}} \int_{-\infty}^{+\infty} e^{is'^2/2} ds' + O\left(\left(\frac{\Omega}{\sqrt{P}}\right)^2\right) \\ &= -e^{i\pi/4} \frac{\sqrt{2\pi}\Omega}{\sqrt{P}} + O\left(\left(\frac{\Omega}{\sqrt{P}}\right)^2\right) \end{aligned} \quad (7)$$

and

$$\begin{aligned} b_2(+\infty) &= 1 - a_2(-\infty) \times i \frac{\Omega}{\sqrt{P}} \int_{-\infty}^{+\infty} e^{-is'^2/2} ds' + O\left(\left(\frac{\Omega}{\sqrt{P}}\right)^2\right) \\ &= 1 + O\left(\left(\frac{\Omega}{\sqrt{P}}\right)^2\right) \end{aligned} \quad (8)$$

for

$$\frac{\Omega}{\sqrt{P}} \ll 1 \quad (9)$$

Thus to the leading order, the transformed state vector $a_2(t = \infty), b_2(t = \infty)$ is

$$\begin{pmatrix} -e^{i\pi/4} \sqrt{2\pi}\Omega/\sqrt{P} \\ 1 \end{pmatrix}$$

and the original vector is

$$\begin{pmatrix} a_1(T \rightarrow \infty) \\ b_1(T \rightarrow \infty) \end{pmatrix} = \begin{pmatrix} -e^{i\pi/4} \sqrt{2\pi} \Omega / \sqrt{P} \lim_{T \rightarrow \infty} e^{-iPT^2/2} \\ 1 \end{pmatrix} \quad (10)$$

4

The Hamiltonian for dipolar coupling is

$$H_1 = \begin{pmatrix} \omega_0/2 & 2\Omega \cos(\omega t) \\ 2\Omega \cos(\omega t) & -\omega_0/2 \end{pmatrix}$$

However, we don't know how to solve this equation analytically. The rotating wave approximation use the following Hamiltonian instead

$$H_2 = \begin{pmatrix} \omega_0/2 & \Omega e^{-i\omega t} \\ \Omega e^{i\omega t} & -\omega_0/2 \end{pmatrix} \quad (11)$$

Using H_2 and the substitution for state vector

$$\begin{aligned} a_1 &= a_2 e^{-i\omega t} \\ b_1 &= b_2 \end{aligned} \quad (12)$$

we get

$$\begin{aligned} i\dot{a}_2 &= (\omega_0/2 - \omega)a_2 + \Omega b_2 \\ i\dot{b}_2 &= \Omega a_2 - \omega_0/2 \end{aligned} \quad (13)$$

The eigen-energy of the corresponding effective Hamiltonian is

$$\lambda_{\pm} = -\frac{\omega}{2} \pm \sqrt{\left(\frac{\omega_0 - \omega}{2}\right)^2 + \Omega^2} \quad (14)$$

The evolution for (a_2, b_2) is

$$\begin{aligned} a_2(t_0 + \Delta) &= \left\{ a_2(t_0) [\cos(A\Delta) - i \sin(A\Delta) \cos \theta] - i b_2(t_0) \sin \theta \sin(A\Delta) \right\} e^{i\omega\Delta/2} \\ b_2(t_0 + \Delta) &= \left\{ -i a_2(t_0) \sin \theta \sin(A\Delta) + b_2(t_0) [i \cos \theta \sin(AT) + \cos(AT)] \right\} e^{i\omega\Delta/2} \end{aligned} \quad (15)$$

where

$$\begin{aligned} A &= \sqrt{\left(\frac{\omega_0 - \omega}{2}\right)^2 + \Omega^2} \\ \cos \theta &= \frac{\omega_0 - \omega}{2A} \\ \sin \theta &= \frac{\Omega}{A} \end{aligned}$$

Thus

$$\begin{aligned} a_1(t_0 + \Delta) &= \left\{ a_1(t_0) [\cos(A\Delta) - i \sin(A\Delta) \cos \theta] - i e^{-i\omega t_0} b_2(t_0) \sin \theta \sin(A\Delta) \right\} e^{-i\omega\Delta/2} \\ b_2(t_0 + \Delta) &= \left\{ -i a_1(t_0) e^{i\omega t_0} \sin \theta \sin(A\Delta) + b_2(t_0) [i \cos \theta \sin(AT) + \cos(AT)] \right\} e^{i\omega\Delta/2} \end{aligned} \quad (16)$$

Let the system evolve from

$$\begin{aligned} a_1(t=0) &= 0 \\ a_2(t=0) &= 1 \end{aligned} \quad (17)$$

for a short period τ , then shut down the coupling Ω for a longer time T , and turn on the coupling again for τ . At the end of the $2\tau + T$ period, the amplitude

$$a_1(2\tau + T) = (-2i)e^{-i\omega(T+2\tau)/2} \sin\theta \sin(A\tau) \left\{ \cos(A\tau) \cos\left(\frac{\omega_0 - \omega}{2}T\right) - \cos\theta \sin(A\tau) \sin\left(\frac{\omega_0 - \omega}{2}T\right) \right\} \quad (18)$$

The probability for the system to be at the excited state $(1, 0)$ is

$$P = 4 \sin^2\theta \sin^2(A\tau) \left\{ \cos(A\tau) \cos\left(\frac{\omega_0 - \omega}{2}T\right) - \cos\theta \sin(A\tau) \sin\left(\frac{\omega_0 - \omega}{2}T\right) \right\}^2 \quad (19)$$

Defining the Rabi frequency

$$\begin{aligned} \Omega_R &= \sqrt{(\omega_0 - \omega)^2 + 4\Omega^2} \\ &= 2A \end{aligned} \quad (20)$$

and introducing the ' $\pi/2$ pulse ' condition of τ for resonance ($\omega_0 = \omega$)

$$\Omega_R(\omega_0 = \omega)\tau = 2\Omega\tau \quad (21)$$

$$= \pi/2 \quad (22)$$

As a check, we expect the probability to be maximized to 1 when $\omega = \omega_0$. Actually it is. For $\Omega \ll |\omega_0 - \omega|$, one has the probability

$$P \approx 16 \frac{\Omega^2}{(\omega_0 - \omega)^2} \sin^2\left(\frac{\omega_0 - \omega}{2}\tau\right) \cos^2\left(\frac{\omega_0 - \omega}{2}T\right) \quad (23)$$

where we have used $A \approx |\omega_0 - \omega|$, $\cos\theta \approx 1$, etc.

5

5-A

For $\Omega_a \neq 0, \Omega_b = 0$, $|b\rangle$ is the dark state.

5-B

For $\Omega_a = 0, \Omega_b \neq 0$, $|a\rangle$ is the dark state.

5-C

Suppose c is the excited states, and a, b are the two ground states. There is no coupling between state a and state b .

Under the rotating wave approximation,

$$H_1 = \begin{pmatrix} E_a & 0 & \Omega_a e^{i\omega_{ac}t} \\ 0 & E_b & \Omega_b e^{i\omega_{bc}t} \\ \Omega_a e^{-i\omega_{ac}t} & \Omega_b e^{-i\omega_{bc}t} & E_c \end{pmatrix}$$

where state vector (a_1, b_1, c_1) , $\omega_{ac} = E_c - E_a$, $\omega_{bc} = E_c - E_b$, $\Omega_{a,b,c} \in \mathbb{R}^+$.

With the substitution

$$\begin{aligned} a_1 &= a_2 e^{-iE_a t} \\ b_1 &= b_2 e^{-iE_b t} \\ c_1 &= c_2 e^{-iE_c t} \end{aligned} \quad (24)$$

the Hamiltonian is simplified to

$$H_2 = \begin{pmatrix} 0 & 0 & \Omega_a \\ 0 & 0 & \Omega_b \\ \Omega_a & \Omega_b & 0 \end{pmatrix}$$

There are three eigenvalues

$$\lambda_0 = 0 \quad (25)$$

$$\lambda_{\pm} = \pm \sqrt{\Omega_a^2 + \Omega_b^2} \quad (26)$$

and the eigenstates

$$\begin{aligned} |\lambda_0 \rangle' &= \frac{1}{\sqrt{\Omega_a^2 + \Omega_b^2}} (\Omega_b |a \rangle' - \Omega_a |b \rangle') \\ |\lambda_{\pm} \rangle' &= \pm \left[\frac{\Omega_a}{\sqrt{2(\Omega_a^2 + \Omega_b^2)}} |a \rangle' + \frac{\Omega_b}{\sqrt{2(\Omega_a^2 + \Omega_b^2)}} |b \rangle' \right] + \frac{1}{\sqrt{2}} |c \rangle' \end{aligned} \quad (27)$$

where the prime means 'after the substitution'.

We define the dark state as the state with no coupling to the excited state c . Thus $|\lambda_0 \rangle'$ is the dark state, and λ_{\pm} are the two bright states.

6 6-A

Qualitatively, for a multi-electron atom, the electrons occupy the single-electron quantum states in the order of increasing energy. The following factors makes the order deviate from the $1s2s2p3s3p3d4s \dots$ order:

- Pauli exclusion principle
- Screening(shielding) effect: The inner electrons repel the outer ones. Thus Electrons in outer orbital states experience a smaller effective charge than the nuclear charge $+Ze$ for Coulomb interaction.

Quantitatively, people find the empirical rule for the dependence of energy on the principle quantum number n and the orbital quantum number l :

$E_{n,l}$ is approximately equal to, and slightly larger than $(n + l/2) \times$ positive constant.

Thus the order for filling up the electronic configurations ($1s, 2s, 2p, \dots$) should be consistent with the increasing order of $n + l/2$.

Readers interested in this can search for "The Aufbau principle".

6-B

An example of the mathematica code is shown below (borrowed from Arnab)

```
SphericalHarmonicY[1, 0, theta, phi]
Y10[theta, phi] := SphericalHarmonicY[1, 0, theta, phi]Conjugate[SphericalHarmonicY[1, 0, theta, phi]]
ParametricPlot3D[Y10[theta, phi]*Sin[theta]*Cos[phi], Y10[theta, phi]*Sin[theta]*Sin[phi], Y10[theta, phi]*Cos[theta],
theta, 0, Pi, phi, 0, 2Pi]
```

6-C

For the LS coupling, nsnp has the following terms: $^1P_1, ^3P_{2,1,0}$. The degeneracies are 3,5,3,1, respectively.

For the JJ coupling, nsnp has the following terms: $(j_1 = 1/2, j_2 = 1/2; j = 1, 0), (j_1 = 1/2, j_2 = 3/2; j = 2, 1)$. The degeneracy is 3,1,5,3, respectively.

When the spin-orbit interaction increases, the states evolves from the LS coupling scheme to the JJ coupling scheme. The correspondence is

$$\begin{aligned} ^1P_1 &\rightarrow (j_1 = 1/2, j_2 = 1/2; j = 1) \\ ^3P_0 &\rightarrow (j_1 = 1/2, j_2 = 1/2; j = 0) \\ ^3P_1 &\rightarrow (j_1 = 1/2, j_2 = 3/2; j = 1) \\ ^3P_2 &\rightarrow (j_1 = 1/2, j_2 = 3/2; j = 2) \end{aligned} \quad (28)$$

The reader can refer to the textbook by Foot (page 88).

7

In the ground state of ${}^6\text{Li}$, the fine structure configuration is ${}^2S_{1/2}$; the total nuclear spin is $I = 1$. There are two hyperfine manifolds and six hyperfine states:

$$\begin{aligned}
 |F = 1/2, m_F = 1/2\rangle &\equiv |1/2, 1/2\rangle_F \equiv |1\rangle \\
 |F = 3/2, m_F = 1/2\rangle &\equiv |3/2, 1/2\rangle_F \equiv |2\rangle \\
 &|1/2, -1/2\rangle_F \equiv |3\rangle \\
 &|3/2, -1/2\rangle_F \equiv |4\rangle \\
 &|3/2, 3/2\rangle_F \equiv |5\rangle \\
 &|-3/2, -3/2\rangle_F \equiv |6\rangle
 \end{aligned} \tag{29}$$

The reason to put them in this order is that $|1\rangle$ and $|2\rangle$, $|3\rangle$ and $|4\rangle$ are coupled respectively in the presence of an external magnetic field.

These states can be expressed in terms of the six $|m_S, m_I\rangle$ basis vectors:

$$\begin{aligned}
 |m_S = -1/2, m_I = 1\rangle_{SI} &\equiv |-1/2, 1\rangle_{SI} \\
 &|1/2, 0\rangle_{SI} \\
 &|-1/2, 0\rangle_{SI} \\
 &|1/2, -1\rangle_{SI} \\
 &|1/2, 1\rangle_{SI} \\
 &|-1/2, -1\rangle_{SI}
 \end{aligned} \tag{30}$$

This order comes from the fact that $|-1/2, 1\rangle_{SI}$ and $|1/2, 0\rangle_{SI}$, $|-1/2, 0\rangle_{SI}$ and $|1/2, -1\rangle_{SI}$ are coupled by the hyperfine interaction. The corresponding eigenstates of H matrix, in the absence of external magnetic field, are $|1\rangle$ to $|6\rangle$, respectively.

The Hamiltonian is

$$\begin{aligned}
 \hat{H} &= a\vec{S} \cdot \vec{I} - \vec{\mu}_e \cdot \vec{B} \\
 &= a/2(\hat{F}^2 - \hat{I}^2 - \hat{S}^2) + 2\mu_B B S_z
 \end{aligned} \tag{31}$$

In order to calculate the matrix element, we write

$$\begin{aligned}
 \hat{S}_x &= 1/2(\hat{S}_+ + \hat{S}_-) \\
 \hat{S}_y &= -i/2(\hat{S}_+ - \hat{S}_-)
 \end{aligned} \tag{32}$$

and similar expressions for \hat{I}_x, \hat{I}_y , where it can be proved that $\hat{S}_+|m_S\rangle = \sqrt{S(S+1) - m_S(m_S+1)}|m_S+1\rangle$ and $\hat{S}_-|m_S\rangle = \sqrt{S(S+1) - m_S(m_S-1)}|m_S-1\rangle$. Thus

$$\hat{H} = \frac{a}{2}(\hat{S}_+\hat{I}_- + \hat{S}_-\hat{I}_+) + a\hat{S}_z\hat{I}_z - \vec{\mu}_e \cdot \vec{B} \tag{33}$$

Thus, in the $|m_S, m_I\rangle$ basis, the Hamiltonian matrix is

$$H_{SI\text{basis}} = \begin{pmatrix} -\frac{a}{2} - \mu_B B & \frac{a}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{a}{\sqrt{2}} & \mu_B B & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu_B B & \frac{a}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{a}{\sqrt{2}} & -\frac{a}{2} + \mu_B B & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{a}{2} + \mu_B B & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{a}{2} - \mu_B B \end{pmatrix} \tag{34}$$

Diagonalizing the matrix, we get the eigenenergies

$$\begin{aligned}
E_1 &= -\frac{a}{4} - \sqrt{(\mu_B B + a/4)^2 + a^2/2} \\
E_2 &= -\frac{a}{4} + \sqrt{(\mu_B B + a/4)^2 + a^2/2} \\
E_3 &= -\frac{a}{4} - \sqrt{(\mu_B B - a/4)^2 + a^2/2} \\
E_4 &= -\frac{a}{4} + \sqrt{(\mu_B B - a/4)^2 + a^2/2} \\
E_5 &= \frac{a}{2} + \mu_B B \\
E_6 &= \frac{a}{2} - \mu_B B
\end{aligned} \tag{35}$$

When $B = 0$, E_1, \dots, E_6 correspond to $|1\rangle_F, \dots, |6\rangle_F$ as eigenstates.

8 8-A

Defining $|S = 1/2, L = 1; m_S, m_L\rangle$ as $|m_S, m_L\rangle_{SL}$, we write the Hamiltonian matrix in the basis of $|1/2, -1\rangle_{SL}$, $|-1/2, 0\rangle_{SL}$, $|1/2, 0\rangle_{SL}$, $|-1/2, 1\rangle_{SL}$, $|1/2, 1\rangle_{SL}$, $|-1/2, -1\rangle_{SL}$:

$$H = \begin{pmatrix} -A/2 & A/\sqrt{2} & 0 & 0 & 0 & 0 \\ A/\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A/\sqrt{2} & 0 & 0 \\ 0 & 0 & A/\sqrt{2} & -A/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & A/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & A/2 \end{pmatrix} \tag{36}$$

where we have used the expression similar to that in problem 7:

$$\begin{aligned}
\hat{H} &= A\vec{S} \cdot \vec{L} \\
&= \frac{A}{2}(\hat{S}_+\hat{L}_- + \hat{S}_-\hat{L}_+) + A\hat{S}_z\hat{L}_z
\end{aligned} \tag{37}$$

The eigenenergies are

$$\begin{aligned}
E_1 &= -A \\
E_2 &= A/2 \\
E_3 &= -A \\
E_4 &= A/2 \\
E_5 &= A/2 \\
E_6 &= A/2
\end{aligned}$$

The corresponding eigenstates are

$$\begin{aligned}
|E_1\rangle &= -\sqrt{2/3}|1/2, -1\rangle_{SL} + \sqrt{1/3}|-1/2, 0\rangle_{SL} = |J = 1/2, m_J = -1/2\rangle \\
|E_2\rangle &= \sqrt{1/3}|1/2, -1\rangle_{SL} + \sqrt{2/3}|-1/2, 0\rangle_{SL} = |J = 3/2, m_J = -1/2\rangle \\
|E_3\rangle &= \sqrt{1/3}|1/2, 0\rangle_{SL} - \sqrt{2/3}|-1/2, 1\rangle_{SL} = |J = 1/2, m_J = 1/2\rangle \\
|E_4\rangle &= \sqrt{2/3}|1/2, 0\rangle_{SL} + \sqrt{1/3}|-1/2, 1\rangle_{SL} = |J = 3/2, m_J = 1/2\rangle \\
|E_5\rangle &= |1/2, 1\rangle_{SL} = |J = 3/2, m_J = 3/2\rangle \\
|E_6\rangle &= |-1/2, -1\rangle_{SL} = |J = 3/2, m_J = -3/2\rangle
\end{aligned} \tag{38}$$

8-B

In order to increase the angular momentum along the z-axis by one unit, we need to have the operator $\hat{J}_+ = \hat{J}_x + i\hat{J}_y = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y + i\hat{z}\hat{p}_x - i\hat{x}\hat{p}_z$. If we choose the beam propagating along z+ direction, then we only need to take care of

$$-i(\hat{x} + i\hat{y})\hat{p}_z = -i(\hat{x} + e^{i\pi/2}\hat{y})\hat{p}_z \quad (39)$$

because this involves the \hat{x} and \hat{y} operators which are used in the dipole interaction.

Comparing the term $\hat{x} + e^{i\pi/2}\hat{y}$ with the interaction $\hat{d} \cdot \hat{E} \propto \hat{x} \cdot \hat{E}$, we see the polarization is

$$\begin{aligned} E_x &= E_0 \\ E_y &= E_0 e^{i\pi/2} \end{aligned} \quad (40)$$

This corresponds to a circular polarization – rotating clockwise when viewed from the z+ direction.

8-C The amplitude ratio is

$$\begin{aligned} & \frac{\langle {}^1S_{1/2}, m_S = 1/2 | \hat{d} \cdot \hat{E} | {}^3P_{3/2}, m_J = 1/2 \rangle}{\langle {}^1S_{1/2}, m_S = 1/2 | \hat{d} \cdot \hat{E} | {}^3P_{1/2}, m_J = 1/2 \rangle} \\ &= \frac{\langle {}^1S_{1/2}, m_S = 1/2 | \hat{d} \cdot \hat{E} | \left[\sqrt{2/3} |m_S = 1/2, m_L = 0\rangle + \sqrt{1/3} |m_S = -1/2, m_L = 1\rangle \right]}{\langle {}^1S_{1/2}, m_S = 1/2 | \hat{d} \cdot \hat{E} | \left[\sqrt{1/3} |m_S = 1/2, m_L = 0\rangle - \sqrt{2/3} |m_S = -1/2, m_L = 1\rangle \right]} \\ &= \frac{\langle {}^1S_{1/2}, m_S = 1/2 | \hat{d} \cdot \hat{E} | \sqrt{2/3} |m_S = 1/2, m_L = 0\rangle}{\langle {}^1S_{1/2}, m_S = 1/2 | \hat{d} \cdot \hat{E} | \sqrt{1/3} |m_S = 1/2, m_L = 0\rangle} \\ &= \sqrt{2} \end{aligned} \quad (41)$$

where the second to last equal mark is due to the fact that the dipole interaction does not change the spin wavefunction.

Thus the intensity (probability) is $\sqrt{2}^2 = 2$

8-D

The eigenstates are 1S_1 , ${}^3P_{0,1,2}$ and 1D_2 .

Due to the Pauli exclusion principle, we can not have the symmetric spin triplet ($S = 1$) and the symmetric orbital states where $L = 0$ or $L = 2$ together; nor can we have the antisymmetric spin singlet ($S = 0$) and the anti-symmetric orbital state $L = 1$ together.

9. Thermal gas in a harmonic potential

A.

Consider a classical thermal gas consists of N particles, the phase space distribution $\rho(p, r) \propto \exp(-H/kT)$. The proportional constant, C , can be determined by

$$N = C \int \exp(-H/kT) d^3 p d^3 r \\ = C(2\pi mkT)^{3/2} \left(\frac{2\pi kT}{m\omega^2}\right)^{3/2}$$

The term in the first parenthesis comes from integrating the momentum space and the later comes from integrating the coordinate space. Thus, the density and momentum distribution is

$$\rho(r) = N \left(\frac{m\omega^2}{2\pi kT}\right)^{3/2} \exp\left(-\frac{1}{2} m\omega^2 r^2\right) \\ \rho(p) = N \left(\frac{1}{2\pi mkT}\right)^{3/2} \exp\left(-\frac{p^2}{2mkT}\right)$$

B.

Repeat the same calculation, but replace the Hamiltonian by $H = \frac{p^2}{2m} + Ar$, the momentum part is still the same but the spatial part has changed.

$$N = C(2\pi mkT)^{3/2} \left(\frac{8\pi k^3 T^3}{A^3}\right)$$

And the density and momentum distribution become

$$\rho(r) = N \left(\frac{A^3}{8\pi k^3 T^3}\right) \exp\left(-\frac{Ar}{2kT}\right) \\ \rho(p) = N \left(\frac{1}{2\pi mkT}\right)^{3/2} \exp\left(-\frac{p^2}{2mkT}\right)$$

10. Scattering in a box

A.

For lowest energy, consider s-wave scattering, the three dimensional Shrodinger equation is

$$u'' - \frac{2m}{\hbar^2}(V - E)u = 0$$

Where $u = r\varphi(r)$ and $V = -d$ for $r < r_0$, $V = 0$ for $r_0 < r < L$, $V = \infty$ for $r > L$.

For a bound state solution, inside the well, $u \propto \sin kr$, where $k = \sqrt{2m/\hbar^2(d + |E|)}$

while outside the well, $u \propto \sinh q(L-r)$, where $q = \sqrt{2m/\hbar^2 |E|}$. By matching the wave functions at r_0 , we have $k \cot kr_0 = -q \coth q(L-r_0) \approx -q \coth qL$ ($L \gg r_0$).

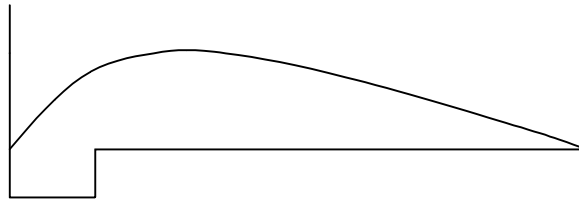
For d is just deep enough to support the first bound state, E will be small. And since we have a large L , the right hand side of the above equation is negative. Thus the above equation will have 1st solution when the LHS turns negative, that is

$$kr_0 \approx \sqrt{2md/\hbar^2} r_0 \geq \pi/2.$$

Thus $d_c = \frac{\pi^2 \hbar^2}{8mdr_0^2}$

B.

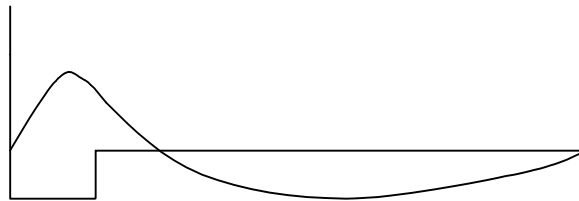
At $r=r_0$, the phase of the inside wave function is close but smaller than $\pi/2$, the system is still unbounded outside the potential well.



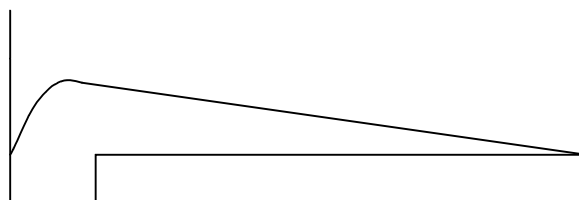
C.

At $r=r_0$, the phase of the inside wave function is close but larger than $\pi/2$. For ground state, the energy is pushed up a little bit such that the system has a first node outside the potential well (the distance can be characterized by the zero energy scattering length a). For bound state, the outside wave function goes to zero in hyperbolic sine form.

Ground state



1st Bound state



D.

When $L/r_0 \rightarrow \infty$, the outside wave function is $u \propto \sin(qr + \delta)$, where δ can be determined by matching the boundary condition at r_0 .

$$\delta = -qr_0 + \arctan\left(\frac{q}{k} \tan kr_0\right)$$

And the scattering length can be calculated by

$$a \underset{q \rightarrow 0}{=} -\frac{\tan \delta}{q} = r_0 \left(1 - \frac{\tan kr_0}{kr_0}\right)$$

Where $k = \sqrt{2md/\hbar^2}$

- i. When $d=0.99d_c$, $k = 0.995\frac{\pi}{2}$ and $a = -80.26r_0$
- ii. When $d= d_c$, $k = \frac{\pi}{2}$, $a = \pm\infty$
- iii. When $d=1.01d_c$, $k = 1.005\frac{\pi}{2}$, $a = 81.85r_0$