1

1-A

The Newton's second law gives

$$\begin{aligned} m_q \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} &= -q\frac{A}{2R^2}\begin{pmatrix} -2x \\ -2y \\ 4z \end{pmatrix} + q\dot{\vec{r}} \times B\hat{z} \\ &= q\frac{A}{R^2}\begin{pmatrix} x \\ y \\ -2z \end{pmatrix} - qB\begin{pmatrix} -\dot{y} \\ \dot{x} \\ 0 \end{pmatrix} \end{aligned} \tag{1}$$

In z direction the ion makes harmonic oscillation with frequency

$$\omega_z = \sqrt{\frac{2Aq}{m_q R^2}} \tag{2}$$

and amplitude z_0.

For the motion in XY plane, we introduce the projection vector $\vec{\rho} = (x, y, 0)$. It satisfies

$$\begin{aligned} \ddot{\vec{\rho}} &= \frac{qA}{m_q R^2}\vec{\rho} - \frac{qB}{m_q}\hat{z} \times \dot{\vec{\rho}} \\ &\equiv \omega_s^2 \vec{\rho} + \vec{\omega}_c \times \dot{\vec{\rho}} \end{aligned} \tag{3}$$

where we introduced

$$\begin{aligned} \omega_s &= \sqrt{\frac{qA}{m_q R^2}} \\ \vec{\omega}_c &= -\frac{qB}{m_q}\hat{z} \\ &= -\omega_c \hat{z} \\ \omega_c &= \frac{qB}{m_q} \end{aligned} \tag{4}$$

1-B

Let $Z \equiv x + iy$. The equation of XY motion can be written as

$$\ddot{Z} = \omega_s^2 Z - i\omega_c \dot{Z} \tag{5}$$

Using the trial solution $Z = e^{i\lambda t}$, we get

$$0 = \lambda^2 + \omega_c \lambda + \omega_s^2 \tag{6}$$

with two roots λ_1, λ_2.

Since we start from $\dot{Z}(0) = \dot{x}(0) + i\dot{y}(0) = 0 + i0 = 0$, the solution must be a linear combination of both $e^{i\lambda_1 t}$ and $e^{i\lambda_2 t}$. If the λ's have non-zero imaginary parts, one of the $e^{i\lambda_i t} (i = 1, 2)$ would diverge with time, which means instability. $\lambda_1 = \lambda_2 \in \mathbb{R}$ also means instability. Thus the two λ's must be real and different. At the same time, it is sufficient for them to be real and different.

Thus the stability condition is

$$\omega_c^2 - 4\omega_s^2 > 0 \tag{7}$$

i.e.

$$\omega_c > 2\omega_s = \sqrt{2}\omega_z \tag{8}$$

or

$$B > \frac{2}{R}\sqrt{\frac{Am_q}{q}} \tag{9}$$

1-C

Using the initial condition $x(0) = x_0, y(0) = 0; \dot{x}(0) = 0, \dot{y}(0) = 0$, we get

$$x + iy = x_0 \left[-\frac{\lambda_2}{\lambda_1 - \lambda_2} e^{i\lambda_1 t} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{i\lambda_2 t} \right] \tag{10}$$

where

$$\omega_{1,2} = -\frac{\omega_c}{2} \pm \sqrt{\frac{\omega_c^2}{4} - \omega_s^2} \tag{11}$$

When $\omega_c >> \omega_s$, we have

$$\begin{aligned} \lambda_1 &\approx -\frac{\omega_s^2}{\omega_c} \\ \lambda_2 &\approx -\omega_c + \frac{\omega_s^2}{\omega_c} \\ |\lambda_2| &>> \omega_s >> |\lambda_1| \end{aligned} \tag{12}$$

When we do the average of x and y over the time scale which is much shorter than $1/|\lambda_1|$, but much longer than $1/|\lambda_2|$, the term proportional to $e^{i\lambda_2 t}$ is averaged out:

$$\begin{aligned} x + iy &\approx x_0 e^{i\lambda_1 t} \\ x &\approx x_0 \cos(\lambda_1 t) \\ y &\approx x_0 \sin(\lambda_1 t) \end{aligned} \tag{13}$$

This corresponds to an effective 2D harmonic potential:

$$\begin{aligned} U_{\text{eff}} &= \frac{1}{2} m_q \lambda_1^2 (x^2 + y^2) \\ &= \frac{m_q \omega_s^4}{2\omega_c^2} (x^2 + y^2) \\ &= \frac{m_q A^2}{2B^2 R^4} (x^2 + y^2) \end{aligned} \tag{14}$$

2

2-A

The radial equation of motion is

$$\begin{aligned} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} &= \begin{pmatrix} -\frac{\partial \phi}{m \partial x} \\ -\frac{\partial \phi}{m \partial y} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{m} \left[-\frac{A}{R^2} \cos(\Omega t) + \frac{A}{Z^2} \right] x \\ \frac{1}{m} \left[\frac{A}{R^2} \cos(\Omega t) + \frac{A}{Z^2} \right] y \end{pmatrix} \\ &\equiv \begin{pmatrix} -\left[a_x - 2q_x \cos(\Omega t) \right] x \\ -\left[a_y - 2q_y \cos(\Omega t) \right] y \end{pmatrix} \end{aligned} \tag{15}$$

where

$$\begin{aligned} a_x &= -\frac{A}{mZ^2} \\ q_x &= -\frac{A}{2mR^2} \\ a_y &= -\frac{A}{mZ^2} \\ q_y &= \frac{A}{2mR^2} \end{aligned} \tag{16}$$

2-B

The stability of the solution of the Mathieu's equation

$$\ddot{x} + (a_x - 2q_x \cos(\Omega t))x = 0 \tag{17}$$

depends on the ratio of two time scales: (1) how fast the potential along a fixed direction (say, along x axis) switch between "trap" and "anti-trap" (2) how fast the particle is dragged out of the center of an anti-trap. To illustrate this point, one does a time scale transformation $\Omega t \equiv \tau$ and introduce the dimensionless quantity $Q = 2q_x/\Omega^2$. The mathieu's equation becomes

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + (a_x/\Omega^2 - Q\cos(\tau))x = 0 \tag{18}$$

Let's ignore a_x/Ω^2 for a moment, and consider the special case

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} - Q\cos(\tau)x = 0 \tag{19}$$

Consider the small Q case. In the period between $\tau = 0$ and $\tau = 2\pi$, the equation is approximately

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} - Q\cos(\tau) < x >= 0 \tag{20}$$

. **With zero initial velocity**, the equation gives

$$x = < x > (1 - Q\cos\tau) \tag{21}$$

This results inspires us to write the overall motion in the form

$$x = x_2(1 - Q\cos\tau) \tag{22}$$

and the equation of motion for x_2 is

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}\tau^2}(1 - Q\cos\tau) + \frac{\mathrm{d}x_2}{\mathrm{d}\tau}(2Q\sin\tau) + x_2 Q^2\cos^2\tau = 0 \tag{23}$$

Because of the special form $1 - Q\cos\tau$, we expect x_2 to be varying much slower than $\cos\tau$ over the period $\tau = 0$ to $\tau = 2\pi$. Then We make an approximation: take the time average over the period between $\tau = 0$ and $\tau = 2\pi$, with x_2 treated as constant. After dropping the average of the fast varying terms: $< \mathrm{d}^2x_2/\mathrm{d}\tau^2 Q\cos\tau >$, $< \mathrm{d}x_2/\mathrm{d}\tau Q\sin\tau >$, and $x_2Q^2\cos(2\tau)$. The equation is reduced to

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}\tau^2} + \frac{Q^2}{2}x_2 = 0 \tag{24}$$

Thus

$$x_2 = X_0\cos(\frac{Q}{\sqrt{2}}\tau + \varphi) \tag{25}$$

and the overall motion is

$$\begin{aligned} x &= X_0\cos(\frac{Q}{\sqrt{2}}\tau + \varphi)(1 - Q\cos\tau) \\ &= X_0\cos(\frac{\sqrt{2}q_x}{\Omega}t + \varphi)(1 - \frac{2q_x}{\Omega^2}\cos(\Omega t)) \\ &\equiv X_0\cos(\omega_c t + \varphi)(1 + q\cos(\Omega t)) \end{aligned} \tag{26}$$

where

$$\begin{aligned} \omega_c &= \frac{\sqrt{2}q_x}{\Omega} \\ q &= -\frac{2q_x}{\Omega^2} \end{aligned} \tag{27}$$

2-C

The ω_c and q are compatible with the physical picture in page 4 of lecture 6. The method here and that of the lecture describe the equation of motion and energy aspects of the same idea: the long time scale motion and the short time scale motion are separable.

3

The Mathieu's equation is

$$\ddot{x} + (a - 2q\cos(2t))x = 0 \tag{28}$$

We think about the equation in two extremes: (1) There exists large anti-trap periods (2) There doesn't exist large anti-trap periods

3-A,B: There exists large anti-trap periods

To gain an intuitive picture of the stable region, we do a simulation for $a = -2$ and different q values. The stable region is:

$$\begin{aligned} 2.39 < &\ q < 2.44 \\ 9.613 < &\ q < 9.632 \\ 23.441 < &\ q < 23.443 \\ 44.0904 < &\ q < 44.0905 \end{aligned} \tag{29}$$

. We see that

- $2|q|$ must be larger than $|a|$.
- The larger q is, the narrower the stable region is.
- The center value of the stable $|q|$ region varies roughly as n^2, but NOT exactly so.

The third rule indicates some resonance structure.

We also provide an intuitive picture as follows. In both (3-A) and (3-B), $|q|$ is large and larger than $|a|/2$, such that we have both large anti-trap periods and trap periods.

In this regime, the ion experiences a potential which varies between a parabolic trap and a very steep anti-parabolic anti-trap. There are four stages for a 'stable' motion. Suppose the ion starts from the center at rest:

- (1)Due to the anti-trap, the particle is accelerated and escapes from the center for quite some distance;
- (2)The ion is decelerated by the trap until it stops;
- (3)The ion continues to be accelerated back towards the center, by the trap;
- (4)The ion is decelerated by the anti-trap until it stop **right at the center**.

In other words, the ion moves periodically (with period T_x). This requires a lot of fine tuning, which is the reason for the narrowness of stable regions. The period T_x is 2π, i.e., 2 times the potential period π.

3-C

Suppose $|q| << 1$. For large and positive a values, the ion is basically doing a harmonic oscillation. The only exception is the cases of resonances. The easiest example is the $a = 1$ case: with proper phase difference, the perturbation $-2q\cos(2t)x$ can be always in the same sign of the harmonic motion velocity, thereby producing the resonance.

Generally, when the harmonic angular frequency $\sqrt{a}$ is an integer multiple of half the perturbation angular frequency $2/2 = 1$, we have a resonance because the effect of the perturbation can accumulate. When the two frequencies do not match, the effect of perturbation can not accumulate, and there is no resonance.

4

4-A

Following the calculations in problem 2, the control parameter is

$$Q = \frac{A}{mR^2\Omega^2} \tag{30}$$

. If we use very large Z to make sure $|a|$ is much less than 1 such that it can be ignored, then we know the critical value of Q , from which the system becomes unstable, to an accuracy much better than 10^{-8}. The relative error for m would be

$$\frac{\Delta m}{m} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(2\frac{\Delta R}{R}\right)^2 + \left(2\frac{\Delta \Omega}{\Omega}\right)^2} \tag{31}$$

Thus

- if we can control two parameters in A, R, and Ω to the relative accuracy of much better than 10^{-8}, then the required relative uncertainty for the third one would be **better than**: 10^{-8} for A varying alone, 5×10^{-9} for R or Ω varying alone. —- In general, all variables have uncertainties; the above upper bounds are necessary but not sufficient condition for achieving 10^{-8} mass spectroscopy.

4-B

With the variance of displacement as the major variable, we put one of the particle sufficiently close to the stable-unstable threshold and still in the stable range. Then the other particle has a Q which is 10^{-8} times larger than the threshold value. The motion amplitude of the first particle remains the same – the envelope is sinusoidal-like. The motion amplitude of the second particle will increase with time – the envelope is exponential-like. Of course we should remember that the two envelopes are similar in small scales, which is due to the very tiny difference in two q values. The smaller the q's difference is, the larger the 'similar region' for the two envelopes is. This indicates that more time is required to distinguish between the two.

First we need to know the threshold value of q better than 10^{-8}. Numerical calculation shows

$$0.908046322509 < q < 0.90804632382 \tag{32}$$

The reader is encouraged to try this in Mathematica. The codes are as follows:

```
q = 0.908046322509
NDSolve[z' '[t] - 2*q*Cos[2t]*z[t] == 0, z[0] == 1, z'[0] == 0, z, t, 0, 90000, MaxSteps − >Infinity]
Plot[Evaluate[z[t] /. %], t, 0, 90000]
  q = 0.90804632382
NDSolve[z' '[t] - 2*q*Cos[2t]*z[t] == 0, z[0] == 1, z'[0] == 0, z, t, 0, 90300, MaxSteps − > Infinity]
Plot[Evaluate[z[t] /. %], t, 0, 90000]
```

Thus we select $q_{\text{stable}} = 0.908046322509$ and $q_{\text{unstable}} = 0.9080463325123$. As shown in the attached figure, the two curves already show difference after a period of $30000 \times \frac{2}{\Omega} = 60000/\Omega$ —- one's amplitude is roughly 2 times as large as that of the other. **Experimentally, we distinguish between the two particles by hearing ONE and ONLY ONE 'click' during the experiment, meaning one and only one particle is out of the trap.** In the calculation, we take the 'factor-of-two' time as an estimate of the 'distinguishing time', which neglects the numerical factors.

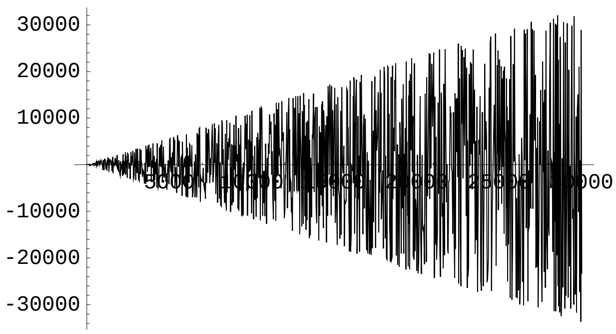
In addition, the sinusoidal or exponential envelopes become apparent after a period of $80000 \times \frac{2}{\Omega} = 160000/\Omega$. Here the dimensionless number 30000 and 80000 correspond to time periods in unit of $2/\Omega$.

```
q = 0.908046322509
NDSolve[{z''[t] - 2 * q * Cos[2 t] * z[t] == 0, z[0] == 1, z'[0] == 0},
 z, {t, 0, 80000}, MaxSteps → Infinity]
Plot[Evaluate[z[t] /. %], {t, 0, 30000}]
```

```
q = 0.9080463325123
NDSolve[{z''[t] - 2 * q * Cos[2 t] * z[t] == 0, z[0] == 1, z'[0] == 0},
 z, {t, 0, 80000}, MaxSteps → Infinity]
Plot[Evaluate[z[t] /. %], {t, 0, 30000}]
```

0.908046

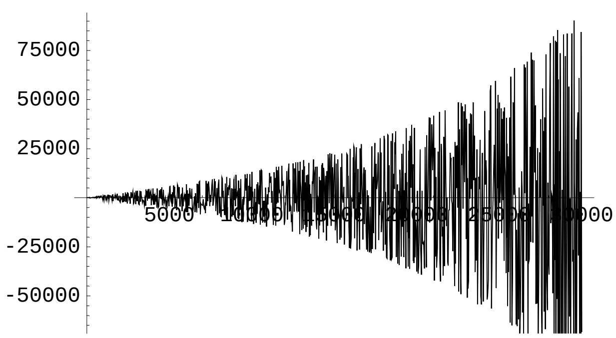
{{z → InterpolatingFunction[{{0., 80000.}}, <>]}}



- Graphics -

0.908046

{{z → InterpolatingFunction[{{0., 80000.}}, <>]}}



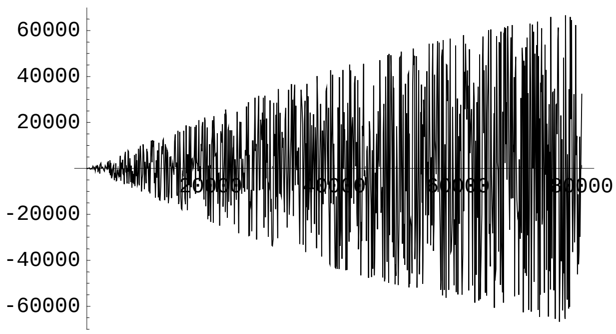
- Graphics -

```
q = 0.9080463225 09
NDSolve[{z''[t] - 2*q*Cos[2 t]*z[t] == 0, z[0] == 1, z'[0] == 0},
 z, {t, 0, 80000}, MaxSteps → Infinity]
Plot[Evaluate[z[t] /. %], {t, 0, 80000}]
```

```
q = 0.9080463325123
NDSolve[{z''[t] - 2*q*Cos[2 t]*z[t] == 0, z[0] == 1, z'[0] == 0},
 z, {t, 0, 80000}, MaxSteps → Infinity]
Plot[Evaluate[z[t] /. %], {t, 0, 80000}]
```

0.908046

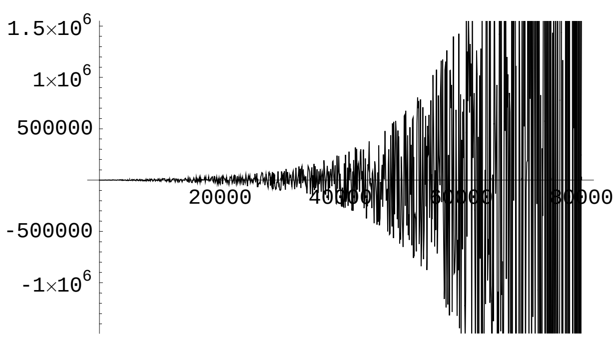
{{z → InterpolatingFunction[{{0., 80000.}}, <>]}}



- Graphics -

0.908046

{{z → InterpolatingFunction[{{0., 80000.}}, <>]}}

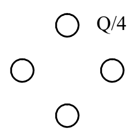


- Graphics -

5. An ion or atom trap with only static E field, no B-field or laser field?

A.

An ion with charge +q in the electric field, ϕ, senses the potential $q\phi$. To confine the ion, we seek for 2D potential minimum. That is to seek for the 2D minimum of ϕ. (In the homework sheet, it asks for maximum, we only need to flip the sign of the charge.)



Two charges +Q/4 in a line create 1-D minimum at the center. Add another two charges +Q/4 symmetrically in a line perpendicular to the first line thus creates 2-D minimum at the center.

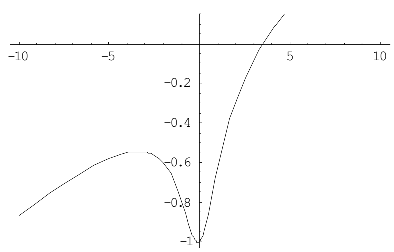
B.

Assume the shortest distance between the charges is a. The combined potential filed at the center along the z direction is

$$\phi(z) = mgz + \frac{Qq}{\sqrt{a^2 + z^2}}$$

To create local minimum, $\frac{d\phi(z)}{dz} = mg - \frac{Qqz}{(a^2 + z^2)^{3/2}} = 0$ should have two real roots.

Which is apparent as long as $mg < \frac{2Qq}{3\sqrt{3}a^2}$.



Plot of the potential at the center of the trap but along the z direction. mg taken to be 0.2 times the critical value.

C.

In real world, $E \sim \frac{Q}{a^2}$ and can be as strong as 100 Mega V/m while mg/q can be several orders of magnitude smaller than this value. The resulting trap size in the z direction (roughly the separation of the two local extremums) will be large. For a realistic trap, the location of the local maximum should at most be roughly at z=a, which requires $\sqrt{\frac{Qq}{mg}} \sim a$, or say Eq/mg should be on the order of unity. This means E is on the order of micro volt per meter. The trap depth is about 0.1 kelvin. It's more than enough to trap cold ions, but the field is too small to control. It's another extreme, and is not realistic.

6

6-A

$$\Delta E(\vec{P},\vec{p},\vec{p}') = \frac{1}{2M}\left[(\vec{P}+\vec{p}-\vec{p}')^2 - \vec{P}^2\right]$$
$$= \frac{1}{2M}\left[2\vec{P}\cdot(\vec{p}-\vec{p}') + (\vec{p}^2 + \vec{p}'^2 - 2\vec{p}\cdot\vec{p}')\right] \quad (33)$$

6-B

If $\vec{p}'$ is undirectional,

$$< \Delta E(\vec{P},\vec{p},\vec{p}') > = \frac{1}{2M}\left[2\vec{P}\cdot\vec{p} + \vec{p}^2 + p'^2\right] \quad (34)$$

where the first two terms represent a incident direction dependent energy transfer caused by the absorption; the second term is the heating effect of the random emission – which is independent of the atomic momentum before the emission.

6-C

If $\vec{p}$ is also undirectional,

$$< \Delta E(\vec{P},\vec{p},\vec{p}') > = \frac{1}{2M}\left[p^2 + p'^2\right] \quad (35)$$

. The first term means the heating due to random absorption. This heating effect also does not depend on the atomic momentum before the absorption.

7

For the absorption spectrum, we need the absolute value of matrix elements of

$$H'_{I,\text{abs}} = \frac{1}{2}\hbar\Omega e^{i\eta(a+a^\dagger)}\sigma^+ \quad (36)$$

where the prime in H means we don't care about the phase. The definition of σ^+ is $\sigma^+|g>=|e>$ and $\sigma^+|e>=0$. $\sigma^- = \sigma^{+\dagger}$.

The absorption spectrum is the series of probability for the system to jump to $|e; n_{\text{vib}}>$:

$$P_{n,\text{abs}} = C\left|< e;n|\frac{1}{2}\hbar\Omega e^{i\eta(a+a^\dagger)}\sigma^+|g;0>\right|^2$$
$$= C\left(\frac{\hbar\Omega}{2}\right)^2\left|< n|e^{i\eta(a+a^\dagger)}|0>\right|^2$$
$$\approx C\left(\frac{\hbar\Omega}{2}\right)^2\left|< n|\frac{i^n\eta^n(a+a^\dagger)^n}{n!}|0>\right|^2$$
$$= C\left(\frac{\hbar\Omega}{2}\right)^2\left|< n|\frac{\eta^n a^{\dagger n}}{n!}|0>\right|^2$$
$$= C\left(\frac{\hbar\Omega}{2}\right)^2\frac{\eta^{2n}}{n!} \quad (37)$$

Using

$$1 = \sum_{n=0}^{\infty} P_{n,\text{abs}}$$
$$= C\left(\frac{\hbar\Omega}{2}\right)^2 e^{\eta^2} \quad (38)$$

we get

$$P_{n,\text{abs}} = \frac{P_{n,\text{abs}}}{1}$$
$$= e^{-\eta^2}\frac{\eta^{2n}}{n!} \tag{39}$$
$$= e^{-E_{\text{recoil}}/(\hbar\omega)}\frac{(E_{\text{recoil}}/(\hbar\omega))^n}{n!},\ n = 0, 1, 2, \cdots \tag{40}$$

where $\eta^2 = E_{\text{recoil}}/(\hbar\omega)$.

7-B

Similar calculation shows exactly the same probability distribution.

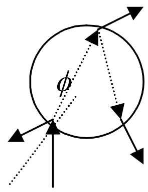
7-C

The energy transfer for an absorption or emission is

$$\Delta E_{n,\text{abs/emi}} = \sum_{n=0}^{\infty} n\hbar\omega P_{n,\text{abs/emi}}$$
$$= \hbar\omega\eta^2$$
$$= E_{\text{recoil}} \tag{41}$$

The average total energy transfer is twice the recoil energy: $2E_{\text{recoil}} = (\hbar k)^2/M_{\text{atom}} = h^2/(M_{\text{atom}}\lambda^2)$. When we take $p' = p$ in 6-C, we find the same total energy transfer as $p^2_{\text{photon}}/M_{\text{atom}}$.

8. Levitation of a glass ball



Let $\vec{k}_0$ be the unit vector representing the direction of the incident light, $\vec{k}_1$ be the vector for the first reflected light and $\vec{k}_{n\geq 2}$ be the vector of (n-1) th transmitted light. Then $\vec{k}_n \cdot \vec{k}_{n+1} = \cos(\pi - 2\phi)$ for n>1, where ϕ is the first refracted angle and $n\sin\phi = \sin\theta$.

The radiation pressure will be

$$\frac{\Delta\vec{F}}{\Delta A} = \frac{I}{C}(\vec{k}_0 - R\vec{k}_1 - \sum_{n=2}^{\infty} T^2 R^{n-2}\vec{k}_n)$$

Where R and T are the reflection and transmission indices and the value depends on the incident angle and the polarization of the light.

For polarization normal to the plane of incidence

$$R_\perp(\theta) = \frac{\sin^2(\theta-\phi)}{\sin^2(\theta+\phi)}, \quad T_\perp(\theta) = \frac{\sin 2\theta \sin 2\phi}{\sin^2(\theta+\phi)}$$

For polarization parallel to the plane of incidence

$$R_{//}(\theta) = \frac{\tan^2(\theta-\phi)}{\tan^2(\theta+\phi)}, \quad T_{//}(\theta) = \frac{\sin 2\theta \sin 2\phi}{\sin^2(\theta+\phi)\cos^2(\theta-\phi)}$$

Integrate over the lower hemisphere of the glass ball gives us the total force. Since I is uniform and the ball is asymuthal symmetric, the total force in the horizontal direction will be zero and the force in the vertical direction will be

$$F_z = \frac{I}{c} r^2 \int d\Omega \cos\theta (\vec{k}_{0z} - R\vec{k}_{1z} - \sum_{n=2}^{\infty} T^2 R^{n-2}\vec{k}_{nz}) = \alpha I r^2 ,$$

where $\alpha = \frac{1}{c}\int d\Omega \cos\theta(\vec{k}_{0z} - R\vec{k}_{1z} - \sum_{n=2}^{\infty} T^2 R^{n-2}\vec{k}_{nz})$,

with R, T taken as the average of two polarization contributions. We thus have

$$\alpha = \frac{2\pi}{c}\int_0^{\pi/2} d\theta \sin\theta\cos\theta (1 + \frac{R_\perp + R_{//}}{2}\cos 2\theta - \sum_{n=1}^{\infty} (-1)^n \frac{T_\perp{}^2 R_\perp{}^{n-1} + T_{//}{}^2 R_{//}{}^{n-1}}{2}\cos(2\theta - 2n\phi))$$

By taking the two terms in the summation can already determine the value to less than 1%, if neglect the summation, it's about 20%.

$\alpha = 0.352\pi / c = 3.69\times 10^9\, s/m$

9
9-A

In the low-intensity limit, the radiation force

$$\begin{aligned}
F &= s_L \hbar k + s_R(-\hbar k) \\
&= -\hbar k \frac{\Gamma}{2}\frac{I}{I_{\text{sat}}}\left(\frac{1}{1+4(\omega_L-\omega_a+\omega_L v/c)^2/\Gamma^2} - \frac{1}{1+4(\omega_L-\omega_a-\omega_L v/c)^2/\Gamma^2}\right) \\
&= -\hbar k \frac{\Gamma}{2}\frac{I}{I_{\text{sat}}}\frac{-1}{\left(1+4\frac{(\omega_L-\omega_a)^2}{\Gamma^2}\right)^2}\frac{4}{\Gamma^2}2(\omega_L-\omega_a)\omega_L\frac{2v}{c} \\
&\approx \hbar k \frac{\Gamma}{2}\frac{I}{I_{\text{sat}}}\frac{16}{\Gamma^2 c}\frac{1}{\left(1+4\frac{(\omega_L-\omega_a)^2}{\Gamma^2}\right)^2}(\omega_L-\omega_a)\omega_L v \\
&\approx \frac{\hbar k\omega_a}{c}\frac{8I}{I_{\text{sat}}}\frac{1}{\left(1+4\frac{(\omega_L-\omega_a)^2}{\Gamma^2}\right)^2}\frac{\omega_L-\omega_a}{\Gamma}v \\
|F/v| &\leq \frac{\hbar k\omega_a}{c}\frac{8I}{I_{\text{sat}}}\frac{3\sqrt{3}}{32} \equiv C_{\text{cool}} \\
\text{'='} &\Leftrightarrow \omega_L-\omega_a = -\Gamma/\sqrt{12}
\end{aligned} \tag{42}$$

where $s_{L,R}$ are the scattering rates for the light coming from the left and right, and $k = 2\pi/\lambda$; in order to achieve damping, the laser should be red de-tuned.

The cooling rate is

$$\begin{aligned}
\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{cool}} &= Fv \\
&= -C_{\text{cool}} v^2
\end{aligned} \tag{43}$$

9-B

Let's treat s_L and s_R not only as the scattering rate, but also as the 'probability'– up to a constant – for the system to absorb an L or R photon in each scattering event. In this case, for a particle (mass M) with initial velocity v, the average kinetic energy **increase** after one scattering event is

$$\begin{aligned}
<\Delta E_k> &= \frac{s_L}{s_L+s_R}\frac{(Mv+\hbar k)^2}{2M} + \frac{s_R}{s_L+s_R}\frac{(Mv-\hbar k)^2}{2M} + \frac{(\hbar k)^2}{2M} - \frac{1}{2}Mv^2 \\
&= \frac{v\hbar k}{s_L+s_R}(s_L-s_R) + \frac{(\hbar k)^2}{M} \\
&= \frac{Fv}{s_L+s_R} + \frac{(\hbar k)^2}{M}
\end{aligned} \tag{44}$$

For small velocities, the first term is negative and corresponds to cooling, and the second term heating. Thus we see even the absorption contributes to the heating process because it still has randomness – either absorbing the L-photon or the R-photon.

The total heating rate is

$$\begin{aligned}
\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{heat, total}} &= \frac{\hbar^2 k^2}{M} \times (s_L+s_R) \\
&\approx \frac{\hbar^2 k^2}{M}\Gamma\frac{I}{I_{sat}}\frac{1}{1+4(\omega_L-\omega_a)^2/\Gamma^2} \\
&= \frac{\hbar^2 k^2}{M}\frac{3\Gamma}{4}\frac{I}{I_{sat}}
\end{aligned} \tag{45}$$

and the heating rate due to absorption or spontaneous emission is the same:

$$\begin{aligned}\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{heat, emi}} &= \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{heat, abs}} \\ &= \frac{\hbar^2 k^2}{M}\frac{3\Gamma}{8}\frac{I}{I_{sat}}\end{aligned} \tag{46}$$

9-C The equilibrium temperature is obtained from

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{cool}} + \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{heat}} = 0 \tag{47}$$

which gives

$$\begin{aligned}T_{\text{equ}} &= \frac{M < v^2 >}{2k_B} \\ &= \frac{\hbar\Gamma}{2\sqrt{3}k_B}\end{aligned} \tag{48}$$

10. Random walk model of Raman cooling

A.

Let's assume in each step each atom is kicked by one photon and has one step random walk. On average, the atoms gain additional energy $\frac{\Delta p^2}{2m}$ and the system temperature kT also increases by this amount. In the mean while, atoms with $E < \frac{\Delta p^2}{2m}$ turn into dark state. In the nth step, there are

$$2/\sqrt{\pi}(\frac{\Delta p^2}{2mkT_n})^{3/2} e^{-\frac{\Delta p^2}{2mkT_n}} \prod_{m=0}^{n-1}(1-2/\sqrt{\pi}(\frac{\Delta p^2}{2mkT_m})^{3/2} e^{-\frac{\Delta p^2}{2mkT_m}}) \approx 2/\sqrt{\pi}(\frac{\Delta p^2}{2mkT_n})^{3/2} \text{ atoms}$$

fall into the dark state. Here, T_n is the temperature after n steps. $kT_{n+1} = kT_n + \frac{\Delta p^2}{2m}$.

$$kT_0 = \frac{N\Delta p^2}{2m}$$

The total portion of atoms fall into dark state is

$$\frac{2}{\sqrt{\pi}}(\frac{\Delta p^2}{2m})^{3/2}\sum_{n=0}^{\infty}(kT_n)^{-3/2} = \frac{2}{\sqrt{\pi}}\sum_{n=N}^{\infty} n^{-3/2} \approx \frac{4}{\sqrt{\pi N}}$$

B.

From A, it takes about N steps to accumulate $1/\sqrt{2}$ of the maximum portion of atoms. This is also the average number of steps for the mostly populated atoms (with momentum $\sqrt{N}\Delta p$) to reach the dark state. The time scale will be N times the photon scattering rate.

C.

The temperature after the Nth step is 2T.