

# 1

The trap depth for the molecules is twice as large as that for the atoms because of the larger molecular polarizability by a factor of two. Taking into account larger molecular mass by a factor of two, it is fair to assume that external energy structures are the same for atoms and molecules. **In the following calculation, we use the canonical ensemble to attack the question. The validity of canonical ensemble is from the constant temperature assumption.**

The partition function of a single atom / molecule is given by

$$Z_a = \sum_{n=0}^{\infty} e^{-\beta E_n} \quad (1)$$

$$\begin{aligned} Z_m &= \sum_{n=0}^{\infty} e^{-\beta E_n + \beta \Delta E} \\ &= e^{\beta \Delta E} Z_a \end{aligned} \quad (2)$$

where  $\beta = 1/(k_B T)$ ;  $\Delta E$  is the absolute value of the binding energy. We are also going to use

$$E_0 = 0 \quad (3)$$

for the atomic ground state energy.

Suppose the atomic / molecular numbers are  $N_a$  and  $N_m$ , respectively. The constraint is

$$N_a + 2N_m = \text{constant} \quad (4)$$

The partition function for the whole system is given by

$$Z = \frac{Z_a^{N_a} Z_m^{N_m}}{N_a! N_m!}$$

The free energy is given by

$$F = -k_B T \log Z \quad (5)$$

Using Stirling's formula,

$$F = -k_B T [N_a \log Z_a + N_m \log Z_m - \log(N_a!) - \log(N_m!)] \quad (6)$$

$$\approx -k_B T [N_a \log Z_a + N_m \log Z_m - (N_a \log N_a - N_a) - (N_m \log N_m - N_m)] \quad (7)$$

Now minimize  $F$ :

$$\frac{dF}{dN_m} = -k_B T \left( \log \frac{Z_m}{Z_a^2} - \log \frac{N_m}{N_a^2} \right) = 0 \quad (8)$$

i.e.

$$\begin{aligned} \frac{N_m}{N_a} &= \frac{N_a Z_m}{Z_a Z_a} \\ &= N_a \frac{e^{-\beta 0}}{Z_a} e^{\beta \Delta E} \\ &= \phi_a e^{\beta \Delta E} \\ &= \phi_a e^{\Delta E / (k_B T)} \end{aligned} \quad (9)$$

where  $\phi_a$  is the ground state population of atoms.

## 2

For classical thermal gas,  $k_B T$  is much larger than the quantum energy spacing. The external energy is

$$\langle E_k + V \rangle = \gamma k_B T \quad (10)$$

at temperature  $T$ .

Although in this problem energy instead of temperature is conserved, we can use the canonical ensemble. This is because different ensembles give the same result at the large particle number limit.

Using the same notations as in Question 1, the constraints are

$$N_0 = N_a + 2N_m \quad (11)$$

$$E_t = N_a \gamma k_B T + N_m (\gamma k_B T - \Delta E) \quad (12)$$

where  $E_t$  is the total energy.

We also use the result from Question 1:

$$\frac{N_m}{N_a} = \frac{N_a}{Z_a} e^{\Delta E / (k_B T)}$$

Thus we get

$$T = \frac{E_t + N_m \Delta E}{\gamma k_B (N_0 - N_m)} \quad (13)$$

Solving the above two equations give the expression for  $N_m$ .

For the  $\Delta E / (k_B T_0) \rightarrow \infty$ , i.e.  $\Delta E \rightarrow \infty$  case, the final temperature is roughly proportional to  $N_m \Delta E$ , which also goes to infinity. The single particle partition function

$$Z_a \sim \left( \frac{k_B T}{\delta} \right)^\gamma \quad (14)$$

where  $\delta$  is the typical energy spacing of the quantum system. This estimation comes from

$$\begin{aligned} \langle E_a \rangle &= - \frac{\partial \log Z_a}{\partial \beta} \\ &= \gamma k_B T \end{aligned} \quad (15)$$

at  $k_B T \gg \delta$ .

The equilibrium condition is simplified to

$$\frac{N_m}{(N_0 - 2N_m)^2} = \left( \frac{\delta}{k_B T} \right)^\gamma \exp \left[ \gamma \frac{N_0 - N_m}{N_m} \right] \quad (16)$$

We see that for any **finite**  $N_m$ , when  $\Delta E \rightarrow 0$ , we have  $T \rightarrow \infty$ , and the right hand side goes to zero, while the left hand side remains finite. This can not be true. Thus  $N_m$  must approach zero, i.e. we have a almost pure atomic sample.

The asymptotic behavior is

$$e^{\gamma N_0 / N_m} \cdot \left( \frac{N_0}{N_m} \right)^{\gamma+1} \approx \left( \frac{\Delta E}{\delta} \right)^\gamma \frac{1}{\gamma^\gamma N_0} \quad (17)$$

i.e.

$$\begin{aligned} \gamma N_0 / N_m &\approx \gamma N_0 / N_m + (\gamma + 1) \log(N_0 / N_m) \\ &\approx \gamma \log(\Delta E / \delta) + \log\left(\frac{1}{\gamma^\gamma N_0}\right) \\ &\approx \gamma \log(\Delta E / \delta) \end{aligned} \quad (18)$$

i.e.

$$\frac{N_0}{N_m} = \log \frac{\Delta E}{\delta} \quad (19)$$

$$N_m \approx \frac{N_0}{\log \frac{\Delta E}{\delta}} \quad (20)$$

where  $\delta$  is the typical energy spacing.

### 3-3-A

$$\begin{aligned}
 H &= \frac{(\vec{p}_1 + \vec{p}_2)^2}{2 \cdot 2m} + \frac{((\vec{p}_1 - \vec{p}_2)/2)^2}{2(m/2)} + \frac{2m\omega^2}{2} \left(\frac{\vec{r}_1 + \vec{r}_2}{2}\right)^2 + \frac{(m/2)\omega^2}{2} (\vec{r}_1 - \vec{r}_2)^2 + V(|\vec{r}_1 - \vec{r}_2|) \\
 &\equiv H_{CM} + H_r
 \end{aligned} \tag{21}$$

where

$$H_{CM} \equiv \frac{(\vec{p}_1 + \vec{p}_2)^2}{2 \cdot 2m} + \frac{2m\omega^2}{2} \left(\frac{\vec{r}_1 + \vec{r}_2}{2}\right)^2 \tag{22}$$

$$H_r \equiv \frac{((\vec{p}_1 - \vec{p}_2)/2)^2}{2(m/2)} + \frac{(m/2)\omega^2}{2} (\vec{r}_1 - \vec{r}_2)^2 + V(|\vec{r}_1 - \vec{r}_2|) \tag{23}$$

### 3-3-B

In the non-interacting regime, with zero temperature, the total energy is  $2 \times 3 \times \hbar\omega/2 = 3\hbar\omega$  for our 3-D system.

### 3-3-C

Let  $V(r) = 0$  for  $r > 0$ .

As we know, the 3-D Schrödinger's equation

$$\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + \frac{1}{2} \mu \omega^2 r^2 \right] \Psi = E\Psi \tag{24}$$

$$r > 0 \tag{25}$$

where  $\mu = m/2$ , is equivalent to

$$\Psi(r) = \frac{u(r)}{\sqrt{2\pi r}} \tag{26}$$

$$\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{1}{2} \mu \omega^2 r^2 \right] u(r) = Eu(r) \tag{27}$$

$$r > 0 \tag{28}$$

The physical meaning of  $u(r)$  is **the reduced 1-D wave-function; the scattering length is defined through  $u(r)$  instead of  $\Psi(r)$ !**

$$\int_0^\infty dr |u(r)|^2 = 1 \tag{29}$$

$$a = -\frac{u(r)}{u'(r)} \tag{30}$$

To get the **asymptotic** behavior of  $E$  for  $a \ll \sqrt{\hbar/(m\omega)}$ , we **don't need to solve the exact eigenstate!** All we need to do is to solve the equation in the  $r \rightarrow 0^+$  regime: the Schrödinger's equation is approximately

$$-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} u(r) \sim Eu(r) \tag{31}$$

for **small but non-zero**  $r$ .

Thus the exact solution has the asymptotic behavior

$$u \sim C \exp\left[-\sqrt{\frac{-2\mu E}{\hbar^2}} r\right] \tag{32}$$

$$r \rightarrow 0^+ \tag{33}$$

At the same time, from the definition of the scattering length:

$$a \equiv -\frac{u(0)}{u'(0)} \tag{34}$$

we get

$$u \sim C' \exp[-r/a] \quad (35)$$

$$r \rightarrow 0^+ \quad (36)$$

Comparing the above two results, we get

$$E = -\frac{\hbar^2}{2\mu a^2} \quad (37)$$

$$= -\frac{\hbar^2}{ma^2} \quad (38)$$

### Remarks

- The 'pure' 3-D harmonic oscillator corresponds to the 1st excited state of the present system when  $a \rightarrow 0$ , not the ground state. See the 'exact solution' section.
- We obtain the ground energy as a function of  $a$  **by approximation**. Qualitatively, the accuracy of the approximation depends on the relative amplitude of the potential energy and the total energy. The smaller  $a$  is, the larger the energy is — the larger range there is where the total energy's absolute  $|E|$  is much larger than the potential energy term  $\mu\omega^2 r^2/2$ . The better the approximation is.

For a rough comparison, we divide  $|E| = \hbar^2/(ma^2)$  by the  $\hbar\omega$  (order of magnitude of the ground potential energy), and let it be much larger than 1. In this way we get  $a \ll \sqrt{\hbar/(m\omega)}$  as a validity criterion for our approximation.

### 3-D

Let's look at the reduced 1-D equation. The lowest possible solution

$$u_0(r) = 2(\mu\omega/(\pi\hbar))^{1/4} \exp[-\mu\omega r^2/(2\hbar)] \quad (39)$$

which corresponds to

$$E = \hbar\omega/2 \quad (40)$$

. We see that  $u_0(0)$  is finite and  $u_0'(0) = 0$ . Thus the scattering length  $|a| = \infty$ .

#### Remarks

This solution does NOT correspond to a solution of a **pure** 3-D harmonic oscillator because of the behavior at  $r = 0$ . However, it does correspond to the ground state of the present system:

$$\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + \frac{1}{2} \mu \omega^2 r^2 \right] \frac{u_0(r)}{\sqrt{4\pi r}} = \frac{\hbar\omega}{2} \frac{u_0(r)}{\sqrt{4\pi r}} - \frac{\hbar^2}{2\mu} \frac{u_0(r)}{\sqrt{4\pi}} \cdot (-4\pi) \delta^{(3)}(\vec{r}) \quad (41)$$

$$(42)$$

i.e.

$$\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + \frac{1}{2} \mu \omega^2 r^2 + V(r) \right] \frac{u_0(r)}{\sqrt{4\pi r}} = \frac{\hbar\omega}{2} \frac{u_0(r)}{\sqrt{4\pi r}} \quad (43)$$

$$V(r) = -\frac{\hbar^2}{2\mu} 4\pi r \delta^{(3)}(\vec{r}) \quad (44)$$

$$= -\frac{\hbar^2}{2\mu} \frac{\delta^+(r)}{r} \quad (45)$$

where  $\delta^+(r)$  is defined as

$$\int_0^\infty dr \delta^+(r) = 1 \quad (46)$$

and we have  $\nabla \frac{1}{r} \cdot \nabla u_0(r) = 0$  at  $r = 0$  due to the fact that  $\nabla f(r) \equiv \vec{0}$  for any  $f(r)$  (because of spherical symmetry).

**3-C,D: An exact solution of the energy spectrum** This part of solution is from: *Foundations of Physics, Vol. 28, No. 4, 1998*. We recalculate the whole process, but the main ideas are exactly the same as in the reference.

Let the interaction term be

$$V = \frac{4\pi\hbar^2 a}{m} \delta^{(3)}(\vec{r}_1 - \vec{r}_2) \frac{\partial}{\partial r} r \quad (47)$$

where  $r = |\vec{r}_1 - \vec{r}_2|$ .

The Schrödinger's equation for the relative motion is

$$\left( -\frac{\hbar^2}{2\mu} \nabla^2 + \frac{1}{2} \mu \omega^2 r^2 + \frac{4\pi\hbar^2 a}{m} \delta^{(3)}(\vec{r}_1 - \vec{r}_2) \frac{\partial}{\partial r} r \right) \Psi = E \Psi \quad (48)$$

where  $\mu = m/2$ .

Let's expand the wave-function with the isotropic eigenstates of the 3-D harmonic oscillator:

$$\Psi(r) = \sum_{n=0}^{\infty} c_n \varphi_n(r) \quad (49)$$

and use

$$\left( -\frac{\hbar^2}{2\mu} \nabla^2 + \frac{1}{2} \mu \omega^2 r^2 \right) \varphi_n(r) = E_n \varphi_n \quad (50)$$

where  $E_n = (2n + 3/2)\hbar\omega$ .

We get

$$\sum_{n=0}^{\infty} c_n (E_n - E) \varphi_n + \frac{4\pi\hbar^2 a}{m} \delta^{(3)}(\vec{r}) \frac{\partial}{\partial r} r \sum_m (c_m \varphi_m) = 0 \quad (51)$$

i.e.

$$c_n (E_n - E) + \frac{4\pi\hbar^2 a}{m} \varphi_n^*(0) \left[ \frac{\partial}{\partial r} r \sum_m (c_m \varphi_m) \right]_{r \rightarrow 0^+} = 0 \quad (52)$$

Thus

$$c_n = A \frac{a \varphi_n^*(0)}{E_n - E} \quad (53)$$

where  $A$  is an  $n$ -independent constant.

We get

$$\frac{4\pi\hbar^2}{m} \left[ \frac{\partial}{\partial r} r \sum_m \frac{\varphi_m^*(0) \varphi_m(r)}{E_m - E} \right]_{r \rightarrow 0^+} = -\frac{1}{a} \quad (54)$$

Using

$$\varphi_n = \pi^{-3/4} \alpha^{3/2} \frac{1}{(L_n^{1/2}(0))^{1/2}} e^{-\alpha^2 r^2/2} L_n^{1/2}(\alpha^2 r^2) \quad (55)$$

where  $\alpha \equiv 1/\sqrt{\hbar/(\mu\omega)} = 1/\sqrt{2\hbar/(m\omega)}$ , we get

$$\frac{4\hbar^2 \alpha^3}{\sqrt{\pi m}} \left[ \frac{\partial}{\partial r} r e^{-\alpha^2 r^2/2} \sum_m \frac{L_m^{1/2}(\alpha^2 r^2)}{E_m - E} \right]_{r \rightarrow 0^+} = -\frac{1}{a} \quad (56)$$

$$\frac{2\hbar\alpha^3}{\sqrt{\pi}m\omega} \left[ \frac{\partial}{\partial r} r e^{-\alpha^2 r^2/2} \sum_m \frac{L_m^{1/2}(\alpha^2 r^2)}{m-\nu} \right]_{r \rightarrow 0^+} = -\frac{1}{a} \quad (57)$$

where  $\nu \equiv \frac{E}{2\hbar\omega} - \frac{3}{4}$ .

For  $n - \nu > 0, n = 0, 1, 2, \dots$ , we have  $\nu < 0$

$$\frac{1}{n-\nu} = \int_0^\infty \frac{dy}{(1+y)^2} \left( \frac{y}{1+y} \right)^{n-\nu-1} \quad (58)$$

thus

$$\begin{aligned} \sum_m \frac{L_m^{1/2}(\alpha^2 r^2)}{m-\nu} &= \int_0^\infty \frac{dy}{\sqrt{1+y}} \left( \frac{y}{1+y} \right)^{-\nu-1} e^{-\alpha^2 r^2 y} \\ &= U(-\nu, 3/2, \alpha^2 r^2) \Gamma(-\nu) \end{aligned} \quad (59)$$

where  $U(a, b, z)$  is the confluent hypergeometric function of the second kind (see <http://mathworld.wolfram.com/ConfluentHypergeometricFunctionoftheSecondKind.html> and *Handbook of Mathematical Functions, page 504 and 505* - which has an online edition <http://www.convertit.com/Go/ConvertIt/Reference/AMS55.ASP>).

We have

$$\begin{aligned} \sum_m \frac{L_m^{1/2}(\alpha^2 r^2)}{m-\nu} &= U(-\nu, 3/2, \alpha^2 r^2) \Gamma(-\nu) \\ &= \Gamma(-\nu)(-\pi) \left[ \frac{M(-\nu, 3/2, \alpha^2 r^2)}{\Gamma(-\nu-1/2)\Gamma(3/2)} - (\alpha^2 r^2)^{-1/2} \frac{M(-\nu-1/2, 1/2, \alpha^2 r^2)}{\Gamma(-\nu)\Gamma(1/2)} \right] \\ &= \Gamma(-\nu)(-\pi) \left[ \frac{1}{\Gamma(-\nu-1/2)\Gamma(3/2)} - (\alpha^2 r^2)^{-1/2} \frac{1}{\Gamma(-\nu)\Gamma(1/2)} + O(r) \right] \\ &= -\frac{\pi\Gamma(-\nu)}{\Gamma(-\nu-1/2)\Gamma(3/2)} + \frac{\sqrt{\pi}}{\alpha r} + O(r) \end{aligned} \quad (60)$$

where

$$M(a, b, z) = 1 + az/b + \frac{(a)_2 z^2}{(b)_2 2!} + \dots + \frac{(a)_n z^n}{(b)_n n!} + \dots \quad (61)$$

$$(a)_n = a(a+1)\dots(a+n-1) \quad (62)$$

$$(a)_0 = 1 \quad (63)$$

Thus

$$\begin{aligned} -\frac{1}{a} &= -\frac{4\hbar\alpha^3\Gamma(-\nu)}{m\omega\Gamma(-\nu-1/2)} \\ &= -\frac{2\alpha\Gamma(-\nu)}{\Gamma(-\nu-1/2)} \end{aligned} \quad (64)$$

i.e.

$$\frac{1/2\alpha}{a} = \frac{\Gamma(-E/(2\hbar\omega) + 3/4)}{\Gamma(-E/(2\hbar\omega) + 1/4)} \quad (65)$$

This holds for  $\nu < 0$ . However, according to the reference *Foundations of Physics, Vol. 28, No. 4, 1998*, the result can be extended to all  $\nu \neq n$  by analytical continuation.

**Molecular regime:**  $a \ll 1/\alpha = \sqrt{\hbar/(m\omega)}$

Numerical calculation gives

$$\frac{1/(2\alpha)}{a} \quad E/(\hbar\omega) \quad (66)$$

$$1.5 \quad -4.47 \quad (67)$$

$$3 \quad -18 \quad (68)$$

$$6 \quad -72 \quad (69)$$

$$12 \quad -288 \quad (70)$$

$$24 \quad -1152 \quad (71)$$

This suggests

$$\begin{aligned}
 E &\approx -\hbar\omega \times 2 \left( \frac{\sqrt{2\hbar/(m\omega)}/2}{a} \right)^2 \\
 &= -\frac{\hbar^2}{ma^2}
 \end{aligned} \tag{72}$$

Indeed when  $z \equiv -E/(2\hbar\omega) + 1/4 \gg 1$  we have

$$\begin{aligned}
 \frac{1/(2\alpha)}{a} &= \log \frac{\Gamma(z + 1/2)}{\Gamma(z)} \\
 &\approx \left[ z \log(z + 1/2) - (z + 1/2) + \frac{1}{2} \log(2\pi) \right] - \left[ (z - 1/2) \log z - z + \frac{1}{2} \log(2\pi) \right] \\
 &= z \log(1 + 1/(2z)) + (1/2) \log z - 1/2 \\
 &\approx 1/2(\log z - \frac{1}{4z}) \\
 &\approx 1/2(\log z + \log(1 - 1/(4z))) \\
 &= 1/2 \log(z - 1/4) \\
 &= 1/2 \log(-E/(2\hbar\omega))
 \end{aligned} \tag{73}$$

Thus

$$\begin{aligned}
 E &\approx -2\hbar\omega \left( \frac{1/(2\alpha)}{a} \right)^2 = -2\hbar\omega \left( \frac{\sqrt{2\hbar/(m\omega)}/2}{a} \right)^2 \\
 &= -\frac{\hbar^2}{ma^2} \text{ for } 0 < a \ll \sqrt{\hbar/(m\omega)}
 \end{aligned} \tag{74}$$

## Infinite scattering length regime

This is equivalent to

$$\Gamma(-E/(2\hbar\omega) + 1/4) = \infty \tag{75}$$

The smallest energy corresponds to

$$-E/(2\hbar\omega) + 1/4 = 0 \tag{76}$$

i.e.

$$E = \frac{\hbar\omega}{2} \tag{77}$$

The total energy is  $2\hbar\omega$ , which is  $-\hbar\omega$  lower than the non-interacting ground energy  $3\hbar\omega$ .

## 4

The wave-function at  $t = 0$  is given by

$$\Psi(\vec{r}_1, \vec{r}_2, t = 0) = \Psi_m(r) \quad (78)$$

$$= \frac{1}{(2\pi)^{3/2}} \int d^3\vec{p} \exp[i\vec{p} \cdot (\vec{r}_1 - \vec{r}_2)] \tilde{\Psi}_m(\vec{p}) \quad (79)$$

In order that  $\Psi_m(r)$  is isotropic in  $\vec{r}$  space, the Fourier component  $\tilde{\Psi}(\vec{p})$  must be isotropic in  $\vec{p}$  space:

$$\tilde{\Psi}_m(\vec{p}) = \tilde{\Psi}_m(p) \quad (80)$$

Besides, in order that the total kinetic energy is  $E$ , we need

$$E = \int d\vec{p} |\tilde{\Psi}_m(p)|^2 \frac{p^2}{m} \quad (81)$$

where we used the normalization

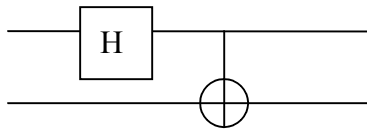
$$\begin{aligned} 1 &= \int d\vec{r} |\Psi_m(r)|^2 \\ &= \int d\vec{p} |\tilde{\Psi}_m(p)|^2 \end{aligned} \quad (82)$$

The wave-function at an arbitrary time is given by

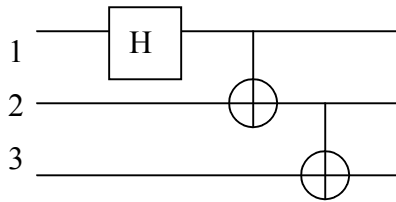
$$\begin{aligned} \Psi(\vec{r}_1, \vec{r}_2, t) &= \frac{1}{(2\pi)^{3/2}} \int d^3\vec{p} \exp \left[ i\vec{p} \cdot (\vec{r}_1 - \vec{r}_2) - i \frac{p^2}{2m} t \cdot 2 \right] \tilde{\Psi}_m(p) \\ &= \frac{1}{(2\pi)^{3/2}} \int d^3\vec{p} \exp \left[ i\vec{p} \cdot (\vec{r}_1 - \vec{r}_2) - i \frac{p^2}{m} t \right] \tilde{\Psi}_m(p) \end{aligned} \quad (83)$$

### 5. N q-bit Greenberger-Horne-Zeilinger state

A.



B.



C.

Adopt the following basis representing the states of q-bits 1, 2, 3.

$$\left( \begin{array}{l} \left( \begin{array}{l} \left( \begin{array}{l} 3 \downarrow \\ 3 \uparrow \end{array} \right) 2 \downarrow \\ ( ) 2 \uparrow \end{array} \right) 1 \downarrow \\ \left( \begin{array}{l} ( ) \\ ( ) \end{array} \right) 1 \uparrow \end{array} \right) \text{ Ex. } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ represents } |\downarrow\downarrow\downarrow\rangle \text{ while } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ represents } |\downarrow\uparrow\uparrow\rangle.$$

Thus, the Hadamard gate applied to q-bit 1 is

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 & I & 0 \\ 0 & I & 0 & I \\ I & 0 & -I & 0 \\ 0 & I & 0 & -I \end{pmatrix}, \text{ followed by the CNOT gate } 1 \oplus 2, \text{ which}$$

$$\text{is } \oplus_{1,2} = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} I & 0 & & \\ 0 & I & & 0 \\ & & 0 & I \\ 0 & & I & 0 \end{pmatrix} \text{ followed by the last CNOT gate } 2 \oplus 3, \text{ which is}$$

$$\oplus_{2,3} = \begin{pmatrix} I & 0 & & \\ 0 & X & & O \\ & & I & 0 \\ O & & 0 & X \end{pmatrix}. \text{ The unitary transformation in the matrix representation is}$$

$$U = \oplus_{2,3} \oplus_{1,2} H = \frac{1}{\sqrt{2}} \begin{pmatrix} I & O & I & O \\ O & X & O & X \\ O & I & O & -I \\ X & O & -X & O \end{pmatrix}.$$

Note that in the above equations we are dealing with 8x8 matrices, we have

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  to expand the matrices to the correct dimension.

D.

Clearly from A to B, adding one q-bit requires adding a CNOT gate to the original GHZ state as a control bit and to the new q-bit as a target bit. The total quantum logic gates needed is just N.

**6****6-A**

The probability that I am NOT poisoned is  $1/2 + 1/2 \times 1/2 = 3/4$

**6-B**

Let  $|edible\rangle = |1\rangle$  and  $|poisonous\rangle = |0\rangle$ .

Use my apple as the control qubit; perform the two-qubit CNOT operation on my apple and each of the  $N$  borrowed edible apples.

If my apple is edible, then the output from the borrowed apples will always be poisonous — poisonous with probability 1.

If my apple is in a superposition, then the output will also be in a superposition — poisonous with probability  $1/2$ .

As long as I borrow sufficiently many edible apples, I can determine the above probability with small enough uncertainty. Thus I can tell whether it is 1 or  $1/2$ , and tell whether my apple is edible or poisonous.

**6-C**

Yes, I can do that. Use my apple as the control qubit; perform the two-qubit CNOT operation on my apple and the borrowed poisonous apple.

If my apple is edible, the output of the borrowed apple will be edible.

If my apple is in a superposition  $(|1\rangle + |0\rangle)/\sqrt{2}$ , then the two-qubit output state will be  $(|11\rangle + |00\rangle)/\sqrt{2}$ .

In both cases, the output of the borrowed apple will have the same edibility as the output of my apple. Thus I use the output of the borrowed apple as the control qubit, and connect the two outputs to a CNOT gate. After that the output of my apple will definitely be  $|0\rangle = |poisonous\rangle$ . As a final step I connect the output of my apple to a Pauli X- single-qubit gate and change it to  $|1\rangle = |edible\rangle$ .

Hence the total energy in the ground state is  $3\hbar\omega/2 + \hbar\omega/2 = 2\hbar\omega$ .

**6-D**

Use the Hadamard single-qubit gate to act on the two apples,

$$H(|1\rangle + |0\rangle)/\sqrt{2} = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} + \frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)/\sqrt{2} = |0\rangle = |poisonous\rangle \quad (84)$$

we can make the outputs definitely poisonous.

## 7. Quantum teleportation

A.

Bell states:

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}}(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle), \quad |\varphi_2\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$$

$$|\varphi_3\rangle = \frac{1}{\sqrt{2}}(|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle), \quad |\varphi_4\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$$

$$|S\rangle_{13} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_3 - |\downarrow\rangle_1|\uparrow\rangle_3)$$

$$|\alpha\rangle_2 = a|\uparrow\rangle_2 + b|\downarrow\rangle_2$$

$$|\phi\rangle = |S\rangle_{13} \otimes |\alpha\rangle_2 = \frac{1}{\sqrt{2}}(a|\uparrow\rangle_1|\uparrow\rangle_2 + b|\uparrow\rangle_1|\downarrow\rangle_2)|\downarrow\rangle_3 - \frac{1}{\sqrt{2}}(a|\downarrow\rangle_1|\uparrow\rangle_2 + b|\downarrow\rangle_1|\downarrow\rangle_2)|\uparrow\rangle_3$$

On the other hand,

$$|\uparrow\uparrow\rangle_{12} = \frac{1}{\sqrt{2}}(|\varphi_1\rangle_{12} - |\varphi_3\rangle_{12}), \quad |\downarrow\uparrow\rangle_{12} = \frac{1}{\sqrt{2}}(|\varphi_2\rangle_{12} + |\varphi_4\rangle_{12})$$

$$|\uparrow\downarrow\rangle_{12} = \frac{1}{\sqrt{2}}(|\varphi_4\rangle_{12} - |\varphi_2\rangle_{12}), \quad |\downarrow\downarrow\rangle_{12} = \frac{1}{\sqrt{2}}(|\varphi_1\rangle_{12} + |\varphi_3\rangle_{12})$$

We have

$$\begin{aligned} |\phi\rangle &= \frac{1}{2}|\varphi_1\rangle_{12}(a|\downarrow\rangle_3 - b|\uparrow\rangle_3) + \frac{1}{2}|\varphi_2\rangle_{12}(a|\uparrow\rangle_3 + b|\downarrow\rangle_3) \\ &\quad - \frac{1}{2}|\varphi_3\rangle_{12}(a|\downarrow\rangle_3 + b|\uparrow\rangle_3) - \frac{1}{2}|\varphi_4\rangle_{12}(a|\uparrow\rangle_3 - b|\downarrow\rangle_3) \\ &= \sum_k \frac{1}{2}|\varphi_k\rangle_{12} U_k |\alpha\rangle_3 \end{aligned}$$

With

$$U_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad U_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

B.

This GHZ state is just the Bell state  $|\varphi_1\rangle$ . The remaining q-bit 3 will be in the state

$U_1|\alpha\rangle_3$ , perform the inverse transformation  $U_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , we can reconstruct state

$|\alpha\rangle$ .

## 8. Simple quantum error correction

A.

Initial state  $|\alpha\rangle \otimes |00\rangle_{12} = a|000\rangle + b|100\rangle$ , after the first CNOT gate,

$|xyz\rangle = |x, x \oplus y, z\rangle$ , we have  $|\ \rangle = a|000\rangle + b|110\rangle$ . After the second CNOT gate,

$|xyz\rangle = |x, y, x \oplus z\rangle$ ,  $|\ \rangle = a|000\rangle + b|111\rangle$ .

B.

In the decoding stage,

$|\ \rangle = a|000\rangle + b|111\rangle \rightarrow a|000\rangle + b|101\rangle \rightarrow a|000\rangle + b|100\rangle = |\alpha\rangle \otimes |00\rangle_{12}$

If error  $\varepsilon|\ \rangle$  is included, we get error term after the decoding

stage  $|error\rangle = \varepsilon|x, x \oplus y, x \oplus z\rangle$ .

1)  $|\ \rangle = |100\rangle$ ,  $|error\rangle = \varepsilon|111\rangle = \varepsilon|1\rangle \otimes |11\rangle_{12}$

2)  $|\ \rangle = |010\rangle$ ,  $|error\rangle = \varepsilon|010\rangle = \varepsilon|0\rangle \otimes |10\rangle_{12}$

3)  $|\ \rangle = |001\rangle$ ,  $|error\rangle = \varepsilon|001\rangle = \varepsilon|0\rangle \otimes |01\rangle_{12}$

C. If the error is  $|\ \rangle = |100\rangle$ , we have one spin flip from the state  $|000\rangle$  (assume the probability of two spin flips, i.e. from state  $|111\rangle$ , is very small). After the decoding stage we have  $|\alpha\rangle \otimes |00\rangle_{12} + \varepsilon|0\rangle \otimes |11\rangle_{12}$ . If we measure  $|00\rangle_{12}$ , the state of the q-bit is (almost) correct  $|\alpha'\rangle = (a - \varepsilon_1)|0\rangle + (b - \varepsilon_2)|1\rangle$ . Another outcome is the measure  $|11\rangle_{12}$ .

The q-bit will be at the state  $|0\rangle$ .

D. We need to look at the outcome of the measurement and tell whether there is error in the q-bit or not. Possible outcomes are  $|00\rangle_{12}$ ,  $|01\rangle_{12}$ ,  $|10\rangle_{12}$ ,  $|11\rangle_{12}$ . When the

later three happens, we have error in the q-bit. However, take  $|01\rangle_{12}$  for example, the

possible one spin flip errors could be  $|\ \rangle' = |001\rangle$  or  $|\ \rangle' = |110\rangle$ . We are not sure whether the q-bit is at the state  $|0\rangle$  or  $|1\rangle$ . We couldn't perform any error correction.

## 9. Bragg diffraction of a Bose-Einstein condensate

A.

Two counter propagating laser beams, the electric field

$$E \propto \sin(kx + \omega t) + \sin(k'x - \omega't)$$

$$= 2 \sin((k + \delta/2c)x - \delta/2t) \cos(-(\delta/2c)x + (\omega + \delta/2)t)$$

The electric field potential the atoms feel is proportional to the time averaging of intensity ( on shorter time scale),

$$\langle I \rangle \propto \langle E^2 \rangle = 2 \sin^2((k + \delta/2c)x - \delta/2t) = 1 - \cos((2k + \delta/c)x - \delta t)$$

Which is a dc term plus a moving “standing wave”, with the velocity

$$v = \frac{\delta}{2k + \delta/c} \approx \frac{\delta}{2k}$$

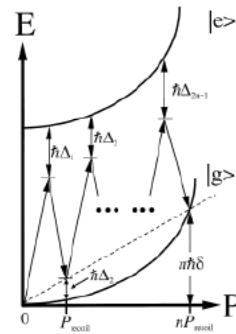
B.

From energy and momentum conservation, from state  $|p = 0\rangle \rightarrow |p = NP_{recoil}\rangle$

$$N\hbar\delta = \frac{(NP_{recoil})^2}{2m} = \frac{4N^2\hbar^2\omega^2}{2mc^2}$$

We get

$$\delta = \frac{2N\hbar\omega^2}{mc^2}, \text{ the resonance condition for } N \text{ recoil.}$$



C.

BEC: atoms diffracted by photon pairs.  $\frac{\hbar}{mc} \cdot \frac{2N\omega^2}{c\delta} = \frac{\hbar}{mc} \cdot \frac{N\omega}{v} = 1$

Laser in crystal: photons diffracted by phonons. From momentum conservation, the Bragg diffraction condition is  $|q_1 - q_2| = K$ , where  $q_1, q_2$  are wave vectors of the lasers and  $K$  is the reciprocal vector generated by the sound wave in the crystal. Compare to matter wave, here we have  $\lambda \frac{2\pi N}{d} = 2 \sin \theta$ , where  $\theta$  is the angle between the laser

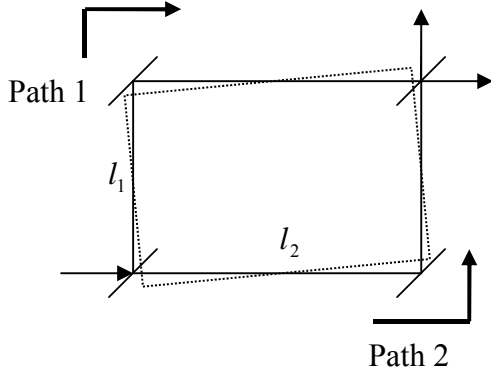
beam and the sound wavefront plane.  $d = \frac{v_{sound}}{f}$  is the spacing between the sound

planes and  $N$  is some integer. Thus we have a similar form  $\lambda \frac{N\omega}{v_{sound}} = 2 \sin \theta$ . The

reason why we have additional factor of 2 here is because photons are relativistic particles while atoms are not.

## 10. Gyroscope based on a Mach-Zehnder interferometer.

A.



To leading order, the time for the light in two different paths to reach the upper right corner is:

$$\text{Path1: } t_1 = \frac{l_1}{c + l_2\Omega/2} + \frac{l_2}{c + l_1\Omega/2} \approx \frac{l_1 + l_2}{c} - \frac{l_1 l_2}{c^2} \Omega$$

$$\text{Path2: } t_2 = \frac{l_2}{c - l_1\Omega/2} + \frac{l_1}{c - l_2\Omega/2} \approx \frac{l_1 + l_2}{c} + \frac{l_1 l_2}{c^2} \Omega$$

Thus the phase difference  $\Phi = \omega(t_1 - t_2) = 2 \frac{A\omega}{c^2} \Omega$ , i.e.  $\frac{d\Phi}{d\Omega} = \frac{2A}{c^2} \omega \propto A$ .

B.

Laser angular frequency is about  $10^{15}$  rad-Hz

$$\text{Spinning revolution } \Phi_{\text{day}} \approx \frac{(10\text{cm})^2}{(3 \times 10^8 \text{ m/sec})^2} 10^{15} \text{ rad/sec} \cdot \frac{2\pi}{86400 \text{ sec}} \approx 10^{-8} \text{ rad}$$

$$\text{Orbital revolution } \Phi_{\text{year}} \approx \Phi_{\text{day}} / 365 \approx 10^{-11} \text{ rad}$$

The resolution should be better than the above value to detect the spinning or orbital revolution of the earth.

C.

For matter wave,  $c$  is to be replaced by the group velocity  $v = \partial\omega / \partial k$  (The particle should be a wave packet instead of a plane wave). Where the relation between  $\omega$  and  $k$

can be obtained from the free particle static Schrödinger equation,  $\frac{\hbar^2 k^2}{2m} = E = \hbar\omega$ .

Thus,

$$\Phi = \frac{A}{v^2} \omega \Omega = \frac{2m}{\hbar} \Omega A$$

D.

For matter wave  $\Phi \approx (3 \cdot 10^5 M \text{ sec}) \times \Omega$ , here, M is the mass number of the atoms used as a probe, we could assume it is of order ten and thus  $\Phi_{matter} \approx (3 \cdot 10^6 \text{ sec}) \times \Omega$ . And from B,  $\Phi_{light} \approx (10^{-4} \text{ sec}) \times \Omega$ . The sensitivity of the matter wave interferometer should be a lot better.