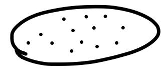
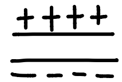
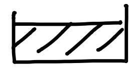
Gross-Pitaevskii Eqn

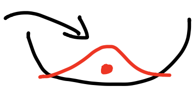
interacting Bosons in the ground state. 

picture : charging a capacitor  $E = \frac{1}{2} QV$

filling a water tank $h \updownarrow$  $E = \frac{1}{2} n h A g$

Let's fill the trap with atoms one by one.

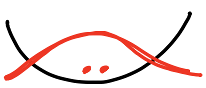
1st atom Schroedinger Eqn



$$\underbrace{\left[\frac{\hbar^2 k^2}{2m} + V(x)\right]}_{H_1} \varphi(x) = E_1 \varphi(x)$$

2nd atom

$$\left[H_1 + H_1 + \underset{\sim}{g\, V(|x_1 - x_2|)}\right] \varphi(x_1, x_2) = E_2\, \varphi(x_1, x_2)$$



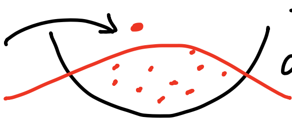
here we assume repulsive int.

Since they are bosons

$$\varphi(x_1, x_2) = \varphi(x_2, x_1)$$

$\vdots$

Nth atom



N-1 atoms inside already

$$H_{N-1}\, \varphi_{N-1}(x_1 \cdots x_{N-1}) = E_{N-1}\, \phi_{N-1}(x_1 \cdots x_{N-1})$$

$$\left[H_{N-1} + H_1 + \sum_{i}^{N-1} V(|x_N - x_i|)\right] \varphi_N(x_1, x_2 \cdots) = E_N \varphi_N$$

Assume $\varphi_N = \varphi_{N-1}\, \phi(x) = \phi_1(x_1)\, \phi_2(x_2) \cdots$

$$\varphi_{N-1}^* \left[H_{N-1} + H_1 + \sum V\right] \varphi_{N-1}\phi = \varphi_{N-1}^* E_N \varphi_{N-1} \phi$$

Integrate over all other N-1 degrees of freedom

$$\Rightarrow E_{N-1}\phi + \left[H_1 + \sum \psi_{N-1}^* V \psi_{N-1}\right] \phi = E_N \phi$$

$$\Rightarrow \left[H_1 + g \sum_i |\phi_i|^2\right] \phi = (E_N - E_{N-1})\phi$$

$$\Rightarrow \left(\frac{\hbar^2 k^2}{2m} + V + g(N-1)|\phi|^2\right)\phi = \mu\phi$$

Introduce condensate wave function $\Psi(x) = \sqrt{N}\,\phi(x)$

s.t. $\int \Psi^*(x)\Psi(x)\,dx = N$ total particle #

$$\Rightarrow \left(\frac{\hbar^2 k^2}{2m} + V + g|\Psi|^2\right)\Psi = \mu\Psi$$

Gross-Pitaevski Eqn.

$\Rightarrow$ Also time-dependent GPE: $i\hbar\partial_t \Psi = \left(\frac{\hbar^2 k^2}{2m} + V + g|\Psi|^2\right)\Psi$

What assumptions have we made?

All atoms occupy exactly the same state.

Wavefunction is separable (not correlated)

Zero temperature: pure-state.

interaction is 2-body & local

Energy of the system.

$$H_N \varphi = E_N \varphi$$

$$H_N = \sum \frac{\hbar^2 k_i^2}{2m} + \sum V(x_i) + \sum_{i<j} V(|x_i - x_j|)$$

$$E_N = \int \varphi^* H_N \varphi \, dx_1 dx_2 \cdots dx_N \qquad \int \psi^* \psi \, dx = N$$

$$= \int \frac{\hbar^2}{2m} |\nabla \psi|^2 dx \quad \text{total kinetic energy}$$

$$+ \int V |\psi|^2 dx \quad \text{total potential energy}$$

$$+ \frac{1}{2} g \int |\psi|^4 dx \quad \text{total int. energy}$$

$\Rightarrow$ energy density $\mathcal{E} = \frac{\hbar^2}{2m} |\nabla \psi|^2 + V |\psi|^2 + \frac{g}{2} |\psi|^4$

In a grand canonical ensemble, free energy $E - \mu N$ minimized.

$\Rightarrow \delta(E - \mu N) = 0$ (variation)

Simplest term: $\delta |\psi|^2 = \varphi \delta \varphi^* + c.c.$

$$\delta |\nabla \varphi|^2 = \nabla \psi \nabla \delta \varphi^* + c.c.$$

$$= \nabla(\nabla \psi \delta \psi^*) - \nabla^2 \delta \psi^* + c.c.$$

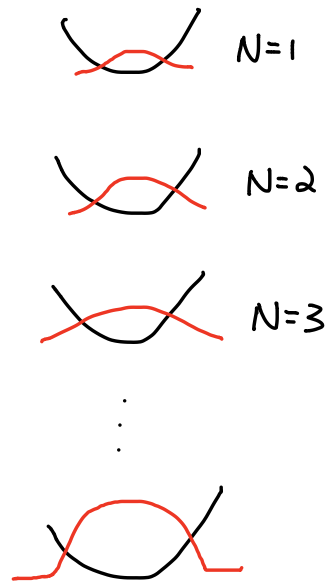
$$\delta |\psi|^4 = \psi^2 \delta \psi^{*2} + c.c. = 2 \psi^2 \psi^* \delta \psi^* + c.c.$$

Put them together:

$$\left(-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi + g |\psi|^2 \psi - \mu \psi\right) \delta \psi^* + c.c. = 0$$

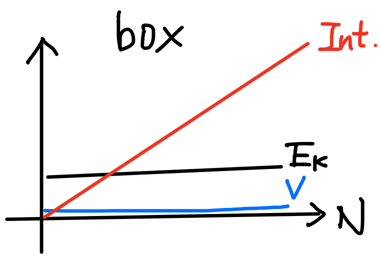
$$\Rightarrow \left(\frac{\hbar^2 k^2}{2m} + V + g |\psi|^2\right) \psi = \mu \psi$$

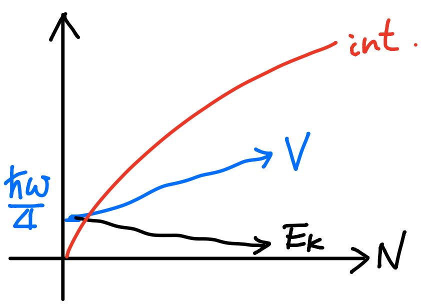
Large particle # limit $N \gg 1$



$$\left[\frac{\hbar^2 k^2}{2m} + V + g|\psi|^2\right]\psi = \mu\psi$$

$\sim \frac{1}{L^2}$ $\sim L^2$ $\sim \frac{N}{L}$





in the large N limit, E_K is not important

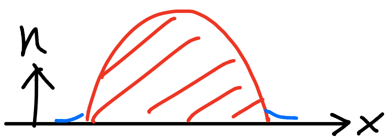
$\Rightarrow$ Thomas-Fermi approximation

$$(V(x) + g|\psi|^2)\psi = \mu\psi$$

$$\Rightarrow n(x) = \frac{1}{g}[\mu - V(x)]$$

In a harmonic trap

$$n(x) = n_0\left[1 - \left(\frac{x}{x_{TF}}\right)^2\right]$$



$$\frac{1}{2} m\omega^2 x_{TF}^2 = \mu.$$