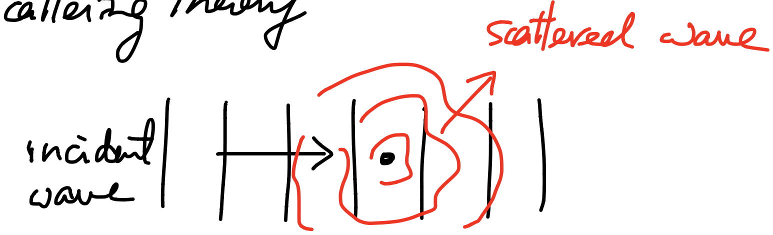
P452 QS&Q   Lec. 4   10/4/2017   Cheng

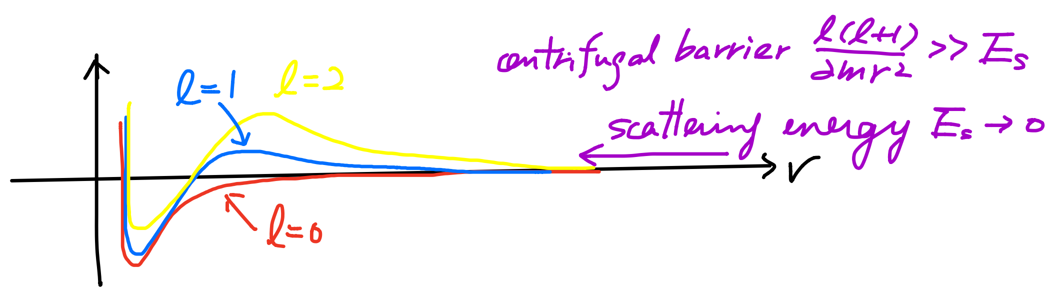
What determine the coupling constant g?

Scattering theory

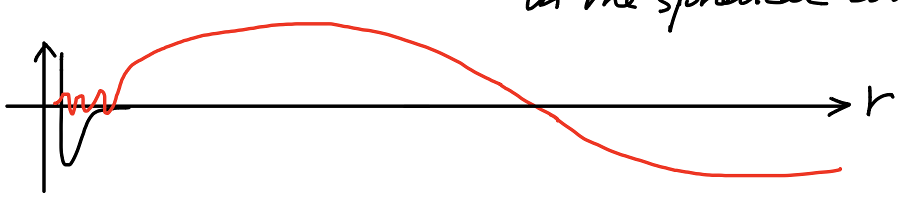


$$\text{wavefunction} = e^{ikz} + f(\theta,\varphi)\frac{e^{ikr}}{r}$$
$$= e^{ikz} + \sum_{lm} f_{lm} Y_l^m(\theta,\varphi)\frac{e^{ikr}}{r}$$

Low temperature limit: only f_{00} is non-zero (s-wave)



Only s-wave (head-on collisions) is left, we have in the spherical coordinate

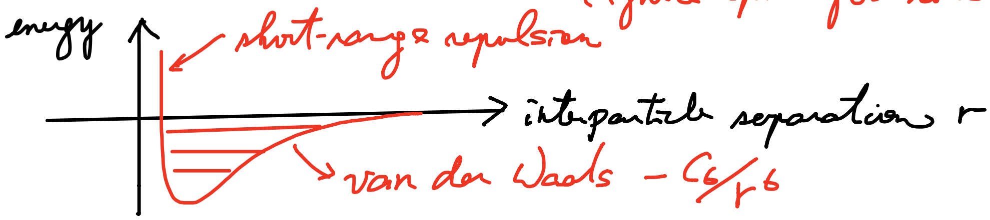


We get $\psi(r) = \frac{e^{-ikr}}{r} - S\,\frac{e^{ikr}}{r}$ Scattering matrix $S \equiv e^{i2\delta}$

incident wave   outgoing wave   scattering phase shift

$= \frac{1}{r} e^{i\delta} \sin(kr+\delta)$

interaction between 2 atoms in the ground state : (ignore spin for now)



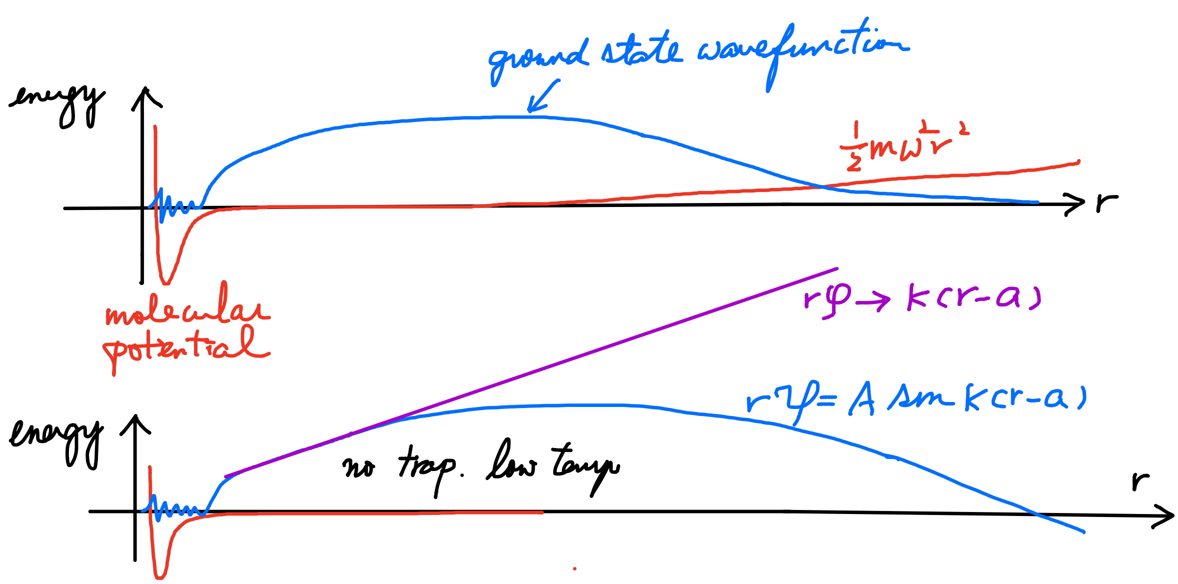
Consider 2 atoms in a trap.

$$H = P_1^2/2m + P_2^2/2m + V(x_1) + V(x_2) + U(|x_1 - x_2|)$$

$$= P^2/2M + p^2/2\mu + V(x_{cm}) + V(r) + U(r)$$

Since trap size is typically $\gg$ molecular potential length

$$H_r = -\frac{\hbar^2}{2\mu}\nabla^2 + U(r) + \cancel{V(r)} \to 0$$



Note that we have assumed s-wave scattering implicitly!

S-wave scattering length: offset of the $l=0$ radial wavefunction.

$$\lim_{r\to 0}\lim_{k\to 0}\frac{r\varphi}{(r\varphi)'}=\lim_{k.r\to 0}\frac{\sin kr+\delta}{k\cos kr+\delta}=\boxed{\lim_{k\to 0}\frac{\tan\delta}{k}=-\overset{\text{scattering length}}{a}}$$

$$\lim_{k\to 0}\varphi\to\frac{A}{r}(r-a)=A(1-\frac{a}{r})$$

$$-\frac{\hbar^2}{2\mu}\nabla^2\varphi=-\frac{\hbar^2}{2\mu}(-Aa)\nabla^2\frac{1}{r}=-\frac{\hbar^2}{m}a4\pi\delta(r)A$$

$$-\frac{\hbar^2}{2\mu}\nabla^2\varphi+\underline{\frac{4\pi a\hbar^2}{m}}\delta(r)\underbrace{[\partial_r r\varphi]}_{\approx\varphi\ \text{why?}}=0.$$

$\Rightarrow$ This is the Schroedinger's eqn with an effective interaction potential $\underline{g\delta(r)}$!

Final result: Coupling constant $g=4\pi a\hbar^2/m$

$$\boxed{i\hbar\partial_t\varphi(x)=\left[P^2/2m+V(x)+\frac{4\pi a\hbar^2}{m}|\varphi|^2\right]\varphi}$$

Additional assumptions we have made:

1. Short-range interactions
2. low-temp scattering (s-wave scattering)

References: Modern Quantum mechanics, JJ Sakurai. Chap. 7
RMP 82. 1225 (2010)