Starting from GP eqn in free space: $i\hbar\partial_t\varphi = (-\frac{\hbar^2}{2m}\nabla^2 + gn)\varphi$

Ground state is $\varphi_0 = \sqrt{n}\, e^{-i\mu t/\hbar}$, chem. potential is $\mu = gn$.

How do we discuss excited states when the eqn is not linear?

At least a moving BEC $\varphi_k = \sqrt{n}\, e^{i(kx-\omega t)}$ is also a solution

here $\hbar\omega = \mu + \frac{\hbar^2 k^2}{2m}$, but $\varphi_0 + \varphi_k$ is generally not a solution.

$$H(|\alpha\rangle + |\beta\rangle) \neq H|\alpha\rangle + H|\beta\rangle$$

* Perturbation to identify any other states that satisfies GPE.

Small perturbation can be described by linear Eqn!!

$i\hbar\partial_t\varphi_0 = H\varphi_0 = \mu\varphi_0$, $\langle\varphi_0|\varphi_0\rangle = N$. now we perturb only few atoms

$\varphi_0(x,t) \rightarrow \varphi_0(x,t) + \delta\varphi(x,t)$.

$$\Rightarrow i\hbar\partial_t\delta\varphi = H\delta\varphi + \delta H\varphi + O(\delta\varphi^2)$$

$$\approx (-\frac{\hbar^2}{2m}\nabla^2 + g\varphi^*\varphi)\delta\varphi + g(\varphi^*\delta\varphi + \varphi\delta\varphi^*)\varphi$$

$$= (-\frac{\hbar^2}{2m}\nabla^2 + 2g\varphi^*\varphi)\delta\varphi + g\varphi^2\delta\varphi^*$$

$$-i\hbar\partial_t\delta\varphi^* = (-\frac{\hbar^2}{2m}\nabla^2 + 2g\varphi^*\varphi)\delta\varphi^* + g\varphi^{*2}\delta\varphi$$

Now we have a coupled pair of linear Eqns.

Observation: if we assume $\delta\varphi = e^{i(kx-\omega t)}$, the eqn will generate $\delta\varphi^* = e^{-i(kx-\omega t)} e^{-i2gnt/\hbar}$, and vice versa, but not any other terms.

Thus a clever choice is $e^{i(kx-\omega t)}e^{-ignt/\hbar}$, which will generate $e^{-i(kx-\omega t)}e^{ignt/\hbar}e^{-2ignt/\hbar}$

Thus we have the trial function

$$\delta\varphi = \left[u\, e^{i(kx-\omega t)} - v^* e^{-i(kx-\omega t)}\right] e^{-ignt/\hbar}$$

plug in $(E_K + 2gn)\delta\varphi + g\varphi^2\delta\varphi^* - i\hbar\partial_t\delta\varphi = 0$

and we know we will have only 2 terms

$e^{-i(kx-\omega t)}e^{-ignt/\hbar}$: $(E_K + 2gn)u - gnv - (\hbar\omega + gn)u$

$e^{i(kx-\omega t)}e^{-ignt/\hbar}$: $(E_K + 2gn)v^* - gnu^* - (-\hbar\omega + gn)v^*$

$$\Rightarrow \begin{pmatrix} E_K + gn - \hbar\omega & -gn \\ -gn & E_K + gn + \hbar\omega \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0$$

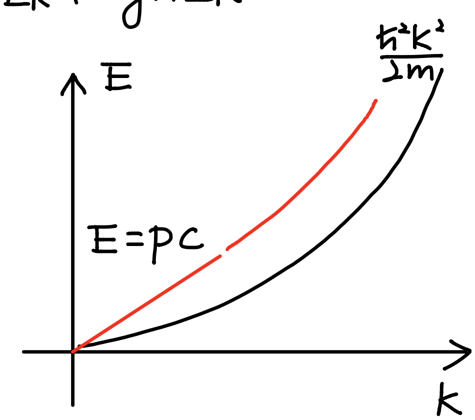
$$\Rightarrow (E_K + gn - \hbar\omega)(E_K + gn + \hbar\omega) = g^2n^2$$

$$\Rightarrow \hbar^2\omega^2 = (E_K + gn)^2 - g^2n^2 = E_K^2 + 2gnE_K$$

$$\Rightarrow \hbar\omega = \sqrt{E_K^2 + 2gnE_K} = \sqrt{\left(\frac{\hbar^2k^2}{2m}\right)^2 + gn\frac{\hbar^2k^2}{m}}$$

low k: $\hbar\omega \to \hbar k c$

high k: $\hbar\omega \to \frac{\hbar^2k^2}{2m} + gn$

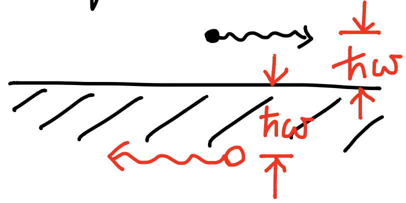


speed of sound $c = \sqrt{gn/m}$

Thus for small perturbation

$$\Psi = e^{-i\mu t/\hbar}\left[\sqrt{n} + u\,e^{i(kx-\omega t)} + v\,e^{-i(kx-\omega t)}\right]$$ is also

a solution as long as $u, v \ll \sqrt{n}$

picture 

if there are u^2 atoms $\rightsquigarrow$
there are v^2 atoms $\leftsquigarrow$

What are u & v?

$$v = \frac{gn}{E_k + gn + \hbar\omega}\,u \Rightarrow \text{low } k:\ v = u$$
$$\text{high } k:\ v \to 0$$

Thus $n(x) = \left[\sqrt{n} + 2u\cos(kx-\omega t)\right]^2$ (low k)

$$= n\left[1 + \frac{4u}{\sqrt{n}}\cos(kx-\omega t)\right]$$

moving ripple $\longrightarrow$

$$n(x) = \left|\sqrt{n} + u\,e^{i(kx-\omega t)}\right|^2$$
$$= n\left(1 + \frac{u}{\sqrt{n}}e^{i(kx-\omega t)}\right)\left(1 + \frac{u}{\sqrt{n}}e^{-i(kx-\omega t)}\right)$$
$$= n + 2\frac{u}{\sqrt{n}}\cos(kx-\omega t)$$

moving ripple $\longrightarrow$

Reference: Bose-Einstein Condensation in Dilute Gases
C. J. Pethick & H. Smith.