Motivation:

$$\langle H \rangle = \int \psi^* H \psi \, dx. \qquad \int \psi^* \psi \, dx = N$$

$$= \int -\frac{\hbar^2}{2m} \psi^* \nabla^2 \psi + \frac{g}{2} |\psi|^4 dx$$

Consider a box of V. we assume ψ can have many momentum components: $\psi = \sum_p a_p V^{-1/2} e^{ipr/\hbar}$

$$E = \sum_p \frac{\hbar^2 k^2}{2m} a_p^* a_p + \frac{g}{2V^2} \int a_{p_1}^* a_{p_2}^* a_{p_3} a_{p_4} e^{-i(p_1+p_2-p_3-p_4)r/\hbar} dx$$

$$= \sum_p \varepsilon_p^0 a_p^* a_p + \frac{g}{2V} \sum_{pp'q} a_{p+q}^* a_{p'-q}^* a_p a_{p'}$$

2nd quantization

$$\hat{H} = \int dr \left[-\hat{\psi}^+ \frac{\hbar^2}{2m} \hat{\psi} + \hat{\psi}^+ V \hat{\psi} + \frac{U_0}{2} \hat{\psi}^+ \hat{\psi}^+ \hat{\psi} \hat{\psi} \right]$$

$$= \sum_p \epsilon_p^0 \hat{a}_p^+ \hat{a}_p + \frac{g}{2V} \sum_{pp'q} \hat{a}_{p+q}^+ \hat{a}_{p'-q}^+ \hat{a}_p \hat{a}_{p'}$$

$\hat{\psi} = \frac{1}{V} \sum \hat{a}_p e^{ipr/\hbar}$, and $|\psi|^4$ should be written as $\psi^{*2}\psi$

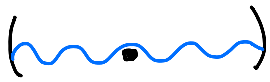
$\hat{a}_p, \hat{a}_p^+$: annihilation and creation operators of an particle with p

A many-body state is labelled as $|n_0, n_1, n_2, n_3 \cdots\rangle$
many-body Fock state

$$\hat{a}_2 |n_0 . n_1 . n_2 \cdots\rangle = \sqrt{n_2} |n_0 . n_1 . n_2 - 1 \cdots\rangle$$
$$\hat{a}_2^+ |n_0, n_1, n_2 \cdots\rangle = \sqrt{n_2+1} |n_0, n_1, n_2+1 \cdots\rangle, \quad \hat{a}^+\hat{a} = \hat{n}$$

Bose commutation relation: $[a_p, a_{p'}^+] = \delta_{pp'}$, others $= 0$

Example: Jaynes-Commines model

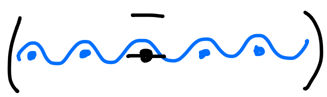


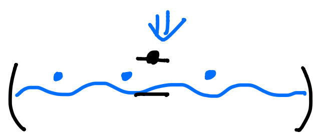
no quantization $H = \frac{1}{2}Kx^2 - f(t)x$

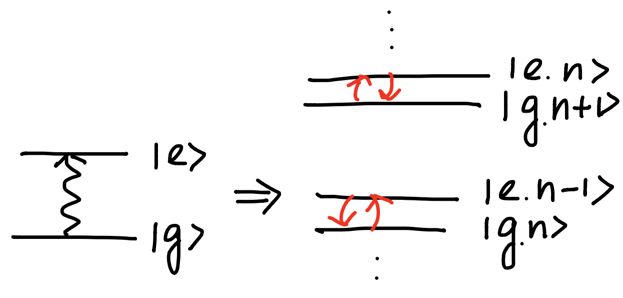
1st quantization $H = \hbar\omega_a \frac{1}{2}\sigma_z - \hat{d}\cdot E$

2nd quantization $H = \hbar\omega_a \frac{1}{2}\sigma_z + \hbar\omega_c a^+ a$

$+ \hbar\Omega \frac{1}{2}(\hat{a}\sigma_+ + \hat{a}^+\sigma_-)$







$$\langle e.n-1| \frac{\hbar\Omega}{2}(\cancel{\hat{a}\sigma_+} + \hat{a}^+\sigma_-)|g.n\rangle$$

$$= \frac{1}{2}\hbar\Omega\sqrt{n}$$

$$\langle g,n| \frac{\hbar\Omega}{2}(\hat{a}\sigma_+ + \hat{a}^+\sigma_-)|e,n-1\rangle$$

$$= \frac{1}{2}\hbar\Omega\sqrt{n}$$

Thus in the $|g.n\rangle.\ |e.n-1\rangle$ subspace we have

$$\begin{bmatrix} E_e + (n-1)\hbar\omega & \sqrt{n}\frac{\hbar\Omega}{2} \\ \sqrt{n}\frac{\hbar\Omega}{2} & E_g + n\hbar\omega \end{bmatrix}$$

We will come back to this in QO class

Back to Bogoliubov

$$\hat{H} = \sum_p \varepsilon_p^0 \hat{n}_p + \frac{g}{2V} \sum_{pp'q} \hat{a}^+_{p+q} \hat{a}^+_{p'-q} \hat{a}_{p'} \hat{a}_p \ , \ \varepsilon_p^0 = p^2/2m$$

Let's consider $|\ \rangle$, where almost all particles are condensed.

$N = N_0 + N_1 + \cdots \approx N_0 \gg N_i \leftarrow$ atoms in the ith excited mode

$$\left.\begin{aligned} \hat{a}_0 |N_0\rangle &= \sqrt{N_0}\, |N_0 - 1\rangle \approx \sqrt{N_0} |N_0\rangle \\ \hat{a}_0^+ |N_0\rangle &= \sqrt{N_0+1}\, |N_0+1\rangle \approx \sqrt{N_0} |N_0\rangle \end{aligned}\right\} \hat{a}_0 . a_0^+ \approx \sqrt{N_0}$$

This is Bogoliubov approximation.

Go straight to interaction term:

$$\frac{g}{2V} \sum \hat{a}^+_{p+q} \hat{a}^+_{p'-q} \hat{a}_{p'} \hat{a}_p = I_1^{0000} + I_2^{\substack{xxoo \\ ooxx}} + I_3^{\substack{oxxo \\ xoox}}$$

Strongest contribution: $p = q = p' = 0$

$$I_1(0000) = \frac{g N_0^2}{2V} = \frac{g}{2V}\Big(N - \sum_{p\neq 0} a_p^+ a_p\Big)^2 = \frac{gN^2}{2V}\Big(1 - \frac{2}{N} \sum_{p\neq 0} a_p^+ a_p\Big)$$

$$I_2\begin{pmatrix} xxoo \\ ooxx \end{pmatrix} = \frac{g N_0}{2V} \sum_{p \neq 0} (a_p^+ a_{-p}^+ + a_p a_{-p})$$

$a_p^+ a_{-p}^+ a_0 a_0$ n $a_0^+ a_0^+ a_p a_p$

$$I_3\begin{pmatrix} oxxo \\ xoox \end{pmatrix} = \frac{g N_0}{2V} 4 \sum_{p\neq 0} a_p^+ a_p$$

$$\left\{ \begin{matrix} a_0^+ a_p^+ a_p a_0 & a_0^+ a_p^+ a_0 a_p \\ a_p^+ a_0^+ a_p a_0 & a_p^+ a_0^+ a_0 a_p \end{matrix} \right.$$

4 combinations

Sum everything together

$$\mathcal{H} = \frac{gN^2}{V} - gn\sum_{p\neq 0} a_p^\dagger a_p + 2gn_0 \sum_{p\neq 0} a_p^\dagger a_p + \sum_{p\neq 0} \varepsilon_p^0 a_p^\dagger a_p + $$

$$+ \sum_{p\neq 0} \frac{gn_0}{2}\left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right)$$

$$= \frac{gN^2}{V} + \sum_{p\neq 0}\left[(\varepsilon_p^0 + gn_0) a_p^\dagger a_p + \frac{gn_0}{2}\left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right)\right]$$

$$= \frac{gN^2}{V} + \sum_{p>0}\left[(\varepsilon_p^0 + gn_0)(a_p^\dagger a_p + a_{-p}^\dagger a_{-p}) + gn_0 (a_p^\dagger a_{-p}^\dagger + a_p a_{-p})\right]$$

this is because $\varepsilon_p^0 = \varepsilon_{-p}^0$

So what do we want?

We want $\mathcal{H} = \sum_k \varepsilon_k^* a_k^\dagger a_k$, can this be done?

This is clear that a_p only couples to a_{-p}

Bogoliubov transformation Bogoliubov (1947)

$$\begin{cases} b_p^\dagger = u\, a_p^\dagger + v\, a_{-p} & , \; u, v \in \mathbb{R} \\ b_{-p} = u\, a_{-p} + v\, a_p^\dagger \end{cases}$$

new operators should be bosonic. $[b_k, b_{k'}^\dagger] = \delta_{kk'}$

$[b_k, b_{k'}] = [b_k^\dagger, b_{k'}^\dagger] = 0$

$\Rightarrow$ constraint: $u^2 - v^2 = 1$

inverse transformation

$$a_p = u\, b_p - v\, b_{-p}^\dagger$$
$$a_p^\dagger = u\, b_p^\dagger - v\, b_{-p}$$

$$h_p \equiv (\varepsilon_p^0 + gn)(a_p^\dagger a_p + a_{-p}^\dagger a_{-p}) + gn(a_p^\dagger a_{-p}^\dagger + a_p a_{-p})$$

$$= (\varepsilon_p^0+gn)\Big[u^2 b_p^\dagger b_p + v^2 b_{-p} b_{-p}^\dagger - uv(b_{-p} b_p + b_p^\dagger b_{-p}^\dagger) +$$

$$u^2 b_{-p}^\dagger b_{-p} + v^2 b_p b_p^\dagger - uv(b_p b_{-p} + b_{-p}^\dagger b_p^\dagger)\Big] +$$

$$gn\Big[u^2 b_p^\dagger b_{-p}^\dagger + v^2 b_{-p} b_p - uv(b_p^\dagger b_p + b_{-p} b_{-p}^\dagger) +$$

$$u^2 b_p b_{-p} + v^2 b_{-p}^\dagger b_p^\dagger - uv(b_p b_p^\dagger + b_{-p}^\dagger b_{-p})\Big]$$

All blue terms can be reduced to $b_k^\dagger b_k$, red terms should be eliminated!!

$$(\varepsilon_p^0+gn)(-2uv)(b_p b_{-p} + b_p^\dagger b_{-p}^\dagger) +$$
$$gn(u^2+v^2)(b_p^\dagger b_{-p}^\dagger + b_p b_{-p}) \equiv 0$$

$$\Rightarrow \frac{2uv}{u^2+v^2} = \frac{gn}{\varepsilon_p^0+gn} \quad (\text{as long as } \varepsilon_p^0 > 0)$$

together with $u^2 - v^2 = 1$. we get $u^2 = \frac{1}{2}\left(\frac{\varepsilon_p^0+gn}{\epsilon_p}+1\right)$, $v^2 = \frac{1}{2}\left(\frac{\varepsilon_p^0+gn}{\epsilon_p}-1\right)$

$\varepsilon_p^0 = p^2/2m$, $\epsilon_p = \sqrt{(\varepsilon_p^0+gn)^2 - g^2n^2}$

Given the above, we

$$(\varepsilon_p^0+gn)\left[(u^2+v^2)(b_p^\dagger b_p + b_{-p}^\dagger b_{-p}) + v^2\right] - 2gnuv(b_p^\dagger b_p + b_{-p}^\dagger b_{-p} + 2)$$

$$= \left[(\varepsilon_p^0+gn)(u^2+v^2) - 2gnuv\right](b_p^\dagger b_p + b_{-p}^\dagger b_{-p}) + (\varepsilon_p^0+gn)v^2 - 2gnuv$$

$$= \varepsilon_p(b_p^\dagger b_p + b_{-p}^\dagger b_{-p}) + \text{const.}$$