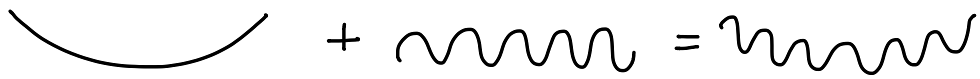
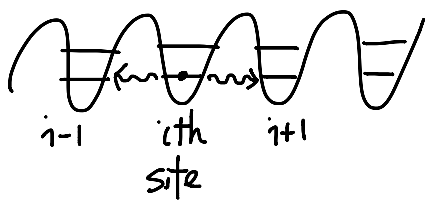
Lecture 9 Optical lattices

1st Step to simulate condensed matter system



At low temperature, atoms occupy the lowest vibrational state (band)



single particles prefer to tunnel (typically)

tunneling: $-t[a_i a_{i+1}^\dagger + a_i a_{i-1}^\dagger +$ all other possibilities

interaction: $\frac{U}{2}\sum n_i(n_i - 1)$ ← # of atom at site i.

Bose-Hubbard model

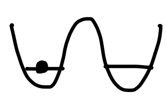
chemical potential

$$H = \underbrace{-t\sum_{\langle i,j\rangle}(a_i^\dagger a_j + a_i a_j^\dagger)}_{E_K} + \underbrace{\frac{U}{2}\sum_i n_i(n_i-1) - \mu\sum_i n_i}_{\text{int.}}$$

When $t \gg U$: delocalization dominates

We expect wave function spreads out.

Example: 1 particle in 2 wells: $E_K = -t(a_1^\dagger a_2 + a_1 a_2^\dagger)$



How do we write wavefunction?

$| \;\; \rangle = |1,0\rangle$ or $|0,1\rangle$ or $A|1,0\rangle + B|0,1\rangle$

$\quad\quad\quad\quad$ LHS $\;$ RHS

Consider $|1,0\rangle$. $\quad \langle E_K\rangle = -t\langle 1,0|(a_1^\dagger a_2 + a_1 a_2^\dagger)|1,0\rangle = 0$

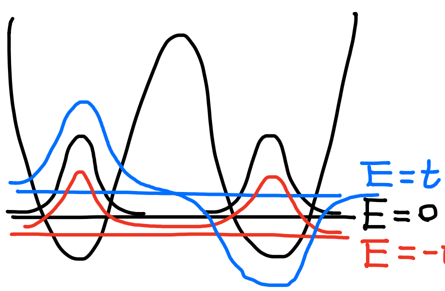
consider $\frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle)$. $\quad a_1 a_2^\dagger \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle) = \frac{1}{\sqrt{2}}|0,1\rangle$

$$\Rightarrow \langle E_K\rangle = -t/2 - t/2 = -t$$

$\frac{1}{\sqrt{2}}(|1,0\rangle - |0,1\rangle)$ $\quad a_1 a_2^\dagger \frac{1}{\sqrt{2}}(|1,0\rangle - |0,1\rangle) = \frac{1}{\sqrt{2}}|0,1\rangle$

$$\Rightarrow \langle E_K\rangle = t/2 + t/2 = t$$

short summary:



high energy → anti-sym. WF

zero energy → localized WF

lowest energy → Symmetric WF

Now consider 1 particle in N sites:
$$E_K = -t \underbrace{\sum_{\langle i,j\rangle}^{N} (a_i^\dagger a_j + h.c.)}_{\text{We have } 2(N-1) \text{ terms}}$$

Let's guess the ground state is fully delocalized:

Guess what's the energy? Each term gives $-\frac{t}{N}$, so $-2t$, yes!

We have the ground state and the highest energy state as

$$|S\rangle = \frac{1}{\sqrt{N}}(|1000\cdots\rangle + |0100\cdots\rangle + |0010\cdots\rangle + \cdots)$$

$$|A\rangle = \frac{1}{\sqrt{N}}(|1000\cdots\rangle - |0100\cdots\rangle + |0010\cdots\rangle - \cdots)$$

How do you do run-of-the-mill calculation?

$$|S\rangle = \frac{1}{\sqrt{N}}\sum_k a_k^+ |0\rangle \qquad \langle S|H|S\rangle = ?$$

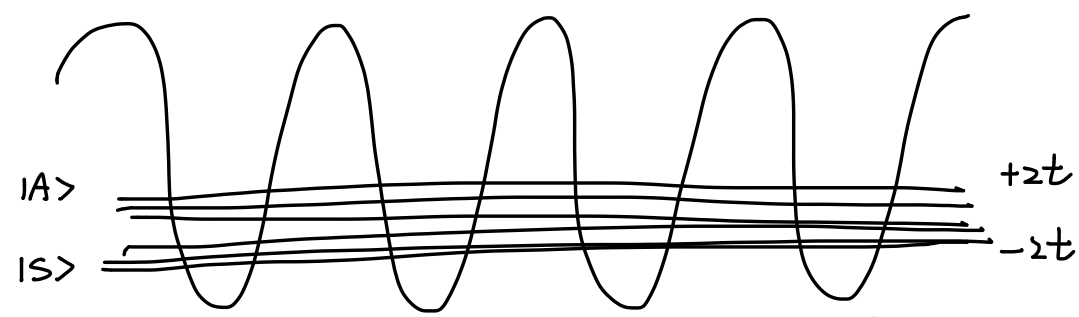
Let's evaluate one term: $-t\, a_i^+ a_j \frac{1}{\sqrt{N}}\sum_k a_k^+ |0\rangle$

$$= -\frac{t}{\sqrt{N}} \sum_k a_i^+ a_j a_k^+ |0\rangle$$

$$= -\frac{t}{\sqrt{N}}\, a_i^+ |0\rangle$$

Now $\langle S|$ from the left: $-\frac{t}{N}\langle 0|\sum_k a_k a_i^+ |0\rangle = -\frac{t}{N}$

So $2(N-1)$ terms gives $\langle S|H|S\rangle = -2t\frac{N-1}{N} \approx -2t$



total N states that forms the ground band when $N \to \infty$

With $m \ll N$ particles $\langle E_K\rangle \approx -2mt$

$$\langle U\rangle \approx \frac{U}{2}\frac{m}{N}$$

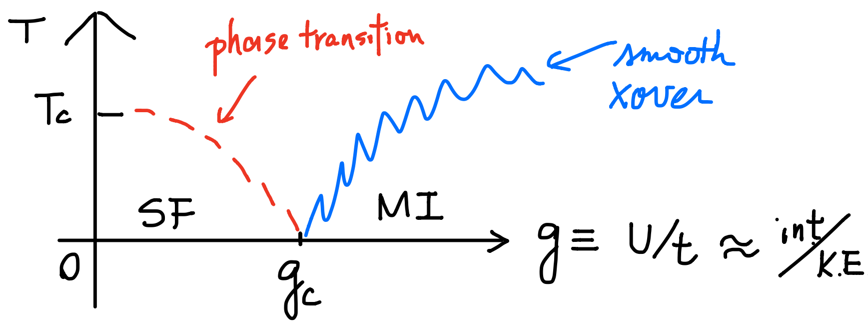
Thus at low density or week int. ground state is always a condensate $|g\rangle \approx \left(\frac{1}{\sqrt{N}}\sum_K a_K^+\right)^N |0\rangle$

Now consider $m = N \gg 1$ particles with some significant $U \approx t$

What happens? Nobody can solve it analytically but we can check the other limit: $U \gg t$.

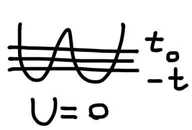
$\text{int} = \frac{U}{2} \sum_i n_i(n_i - 1) = 0$ if $n_i = 1$, this must be the ground state

$$|g\rangle \xrightarrow[U \gg t]{} a_1^\dagger a_2^\dagger a_3^\dagger \cdots |0\rangle = \prod_i a_i^\dagger |0\rangle \quad \text{(Mott insulator)}$$



Can we tell what's going on near g_c? What's the value of g_c?

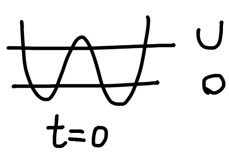
Let's go back to 2 sites, now with 2 particles



$|A\rangle$
$|L\rangle . |R\rangle$
$|S\rangle$

What are the possible configurations?

$|20\rangle$
$|11\rangle$
$|02\rangle$



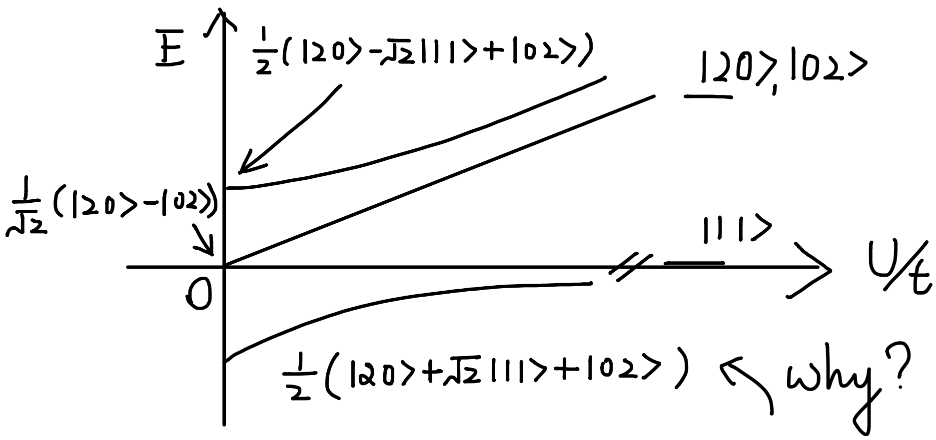
$|20\rangle \alpha |02\rangle$
$|11\rangle$

in general $|\varphi\rangle = \begin{pmatrix} \varphi_{20} \\ \varphi_{11} \\ \varphi_{02} \end{pmatrix}$

$$H|\ \rangle = E_K|\ \rangle + \text{int}|\ \rangle = E|\ \rangle$$

$$E_K = -t\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \qquad \text{int} = \begin{bmatrix} U & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & U \end{bmatrix}$$

$$H|\,\rangle = \begin{bmatrix} U & -2t & 0 \\ -2t & 0 & -2t \\ 0 & -2t & U \end{bmatrix} |\,\rangle = E|\,\rangle$$



$$\hat{S}\,\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)\otimes\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$$

$$= \hat{S}\,\frac{1}{2}(|02\rangle+|20\rangle+|11\rangle+|11\rangle)$$

$$= \frac{1}{2}(|02\rangle+|20\rangle+\sqrt{2}|11\rangle)$$