HW5 Solution

2A. $\Psi_R = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} \Psi \equiv U\Psi$

$$\Rightarrow i\hbar\partial_t U^{-1}\Psi_R = H U^{-1}\Psi_R$$

Note that $U\partial_t U^{-1} = U(-i\frac{\omega}{2}\sigma_z U^{-1} + U^{-1}\partial_t) = i\frac{\omega}{2}\sigma_z + \partial_t$

$$\Rightarrow i\hbar\partial_t\Psi_R = U(H - \frac{\hbar\omega}{2}\sigma_z)U^{-1}\Psi_R \equiv H_R\Psi_R$$

$$\Rightarrow H_R = U\left[\frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x(e^{i\omega t}+e^{-i\omega t}) - \frac{\hbar\omega}{2}\sigma_z\right]U^{-1}$$

$$= \frac{\hbar}{2}\left[(\omega_0-\omega)\sigma_z + \Omega(e^{i\omega t}+e^{-i\omega t})e^{-i\omega t}\sigma_x\right]$$

$\approx \frac{\hbar}{2}(-\Delta\sigma_z + \Omega\sigma_x)$, ignoring the counter rotating term: $e^{-2i\omega t}$

B. $i\hbar\dot{\rho} = [H,\rho]$. $\rho_{ij} = \langle i|\Psi_R\rangle\langle\Psi_R|j\rangle$

$$= \frac{\hbar}{2}(-\Delta[\sigma_z,\rho] + \Omega[\sigma_x,\rho])$$

$$[\sigma_z,\rho] = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\rho_{11}&\rho_{12}\\\rho_{21}&\rho_{22}\end{bmatrix} - \begin{bmatrix}\rho_{11}&\rho_{12}\\\rho_{21}&\rho_{22}\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}\rho_{11}&\rho_{12}\\-\rho_{21}&-\rho_{22}\end{bmatrix} - \begin{bmatrix}\rho_{11}&-\rho_{12}\\\rho_{21}&-\rho_{22}\end{bmatrix} = -2\begin{bmatrix}0&\rho_{12}\\\rho_{21}&0\end{bmatrix}$$

$$[\sigma_x,\rho] = \begin{bmatrix}\rho_{21}&\rho_{22}\\\rho_{11}&\rho_{12}\end{bmatrix} - \begin{bmatrix}\rho_{12}&\rho_{11}\\\rho_{22}&\rho_{21}\end{bmatrix} = \begin{bmatrix}\rho_{21}-\rho_{12}&\rho_{22}-\rho_{11}\\\rho_{11}-\rho_{22}&\rho_{12}-\rho_{21}\end{bmatrix}$$

$$\Rightarrow i\begin{bmatrix}\dot\rho_{11}&\dot\rho_{12}\\\dot\rho_{21}&\dot\rho_{22}\end{bmatrix} = \begin{bmatrix}\frac{\Omega}{2}(\rho_{21}-\rho_{12}) & \rho_{12}\Delta+\frac{\Omega}{2}(\rho_{22}-\rho_{11})\\ \rho_{21}\Delta+\frac{\Omega}{2}(\rho_{11}-\rho_{22}) & \frac{\Omega}{2}(\rho_{12}-\rho_{21})\end{bmatrix}$$

C. We need $\Delta=0$. and thus

$i\dot\rho_{11} = \frac{\Omega}{2}(\rho_{21}-\rho_{12})$. $i(\dot\rho_{21}-\dot\rho_{12}) = \Omega(\rho_{11}-\rho_{22})$

$$\Rightarrow -\ddot\rho_{11} = \frac{\Omega}{2}\Omega(2\rho_{11}-1) = \Omega^2(\rho_{11}-\tfrac{1}{2})$$

$\Rightarrow$ ρ_{11} oscillates as $\cos\Omega t \Rightarrow \Omega t_\pi = \pi$.

D. $\dot\rho_{11} = i\frac{\Omega}{2}(\rho_{12}-\rho_{21}) - \Gamma\rho_{11}$

$\dot\rho_{12} = -i\Delta\rho_{12} + i\frac{\Omega}{2}(\rho_{11}-\rho_{22}) - \frac{\Gamma}{2}\rho_{12}$

$\Rightarrow$ $\rho_{11} = \frac{i\Omega}{2\Gamma}(\rho_{12}-\rho_{21})$

$(i\Delta+\Gamma/2)\rho_{12} = i\frac{\Omega}{2}(\rho_{11}-\rho_{22})$

$(-i\Delta+\Gamma/2)\rho_{21} = -i\frac{\Omega}{2}(\rho_{11}-\rho_{22})$

$\Rightarrow$ $(\Delta^2+\Gamma^2/4)\rho_{12} = i\frac{\Omega}{2}(\rho_{11}-\rho_{22})(-i\Delta+\Gamma/2)$

$(\Delta^2+\Gamma^2/4)\rho_{21} = -i\frac{\Omega}{2}(\rho_{11}-\rho_{22})(i\Delta+\Gamma/2)$

$\Rightarrow \rho_{12}-\rho_{21} = (\Delta^2+\Gamma^2/4)^{-1}\frac{\Omega}{2}(\rho_{11}-\rho_{22})\,i\Gamma$

$$\Rightarrow \rho_{11} = \left(\frac{\Omega}{2}\right)^2(\Delta^2+\Gamma^2/4)^{-1}(1-2\rho_{11}) = \frac{\frac{1}{2}\frac{\Omega^2}{2}(\Delta^2+\Gamma^2/4)^{-1}}{1+\frac{\Omega^2}{2}(\Delta^2+\Gamma^2/4)^{-1}} = \frac{1}{2}\,\frac{\Omega^2/2}{\Omega^2/2+\Delta^2+\Gamma^2/4}$$

$$= \frac{1}{2}\,\frac{2\Omega^2/\Gamma^2}{1+4\Delta^2/\Gamma^2+2\Omega^2/\Gamma^2} = \frac{1}{2}\,\frac{P}{1+P}.$$

E. Scattering rate $S = \Gamma\rho_{11}$, $\sigma = S\hbar\omega/I$

F. Max σ is reached in the limit of low intensity and $\Delta = 0$. $\Rightarrow$

$$\sigma_m = 3\lambda^2/2\pi = S_m\hbar\omega/I = \hbar\omega\frac{\Gamma}{2}\frac{1}{I_s} \Rightarrow I_s = \frac{\hbar\omega\Gamma}{2}\frac{2\pi\omega^2}{3c^2} = \frac{\pi\Gamma\hbar\omega^3}{3c^2}$$

The ω^3 dependence is an important result of Rayleigh scattering, and sky is blue.

G. A more rigorous approach is to solve the optical Bloch Eqn.

We may also just estimate from perturbation

$$\rho_{11} = \frac{1}{2}[1-\cos\Omega t]$$

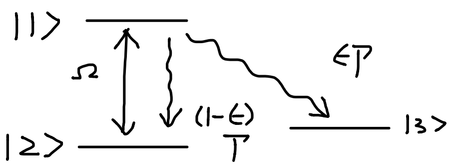
when Ωt reaches π, $\langle\rho_{11}(t_\pi)\rangle = 1/2$ and we may estimate a fraction of $e^{-\Gamma t_\pi}$

will decay. Thus we have $\frac{1}{2}e^{-\Gamma\pi/\Omega} \le \frac{1}{10} \Rightarrow \Omega \ge \frac{\Gamma\pi}{\ln 5}$

$$I/I_s = 2\Omega^2/\Gamma^2 > 2\left(\frac{\pi}{\ln 5}\right)^2 \Rightarrow I > 2\left(\frac{\pi}{\ln 5}\right)^2 I_{sat} = 2\left(\frac{\pi}{\ln 5}\right)^2\frac{\pi}{3}\frac{\Gamma\hbar\omega^3}{c^2}$$

3. Optical pumping

$$H_R = \frac{\hbar}{2}[-\Delta\sigma_z + \Omega\sigma_x] \oplus E_3$$



Since no radiative coupling between $1\leftrightarrow3$ & $2\leftrightarrow3$, we can use the result from 2.

$$\dot\rho_{11} = i\frac{\Omega}{2}(\rho_{12}-\rho_{21}) - \Gamma\rho_{11}$$

$$\dot\rho_{12} = -i\Delta\rho_{12} + i\frac{\Omega}{2}(\rho_{11}-\rho_{22}) - \frac{\Gamma}{2}(1-\epsilon)\rho_{12}$$

$$\dot\rho_{33} = \Gamma\varepsilon\rho_{11}$$

The decay terms are added by hand.

For $\epsilon \ll 1$, we can use the result from 2. $\frac{\rho_{11}}{\rho_{11}+\rho_{22}} = \frac{1}{2}\frac{P}{1+P} \Rightarrow \rho_{11} = (1-\rho_{33})\frac{1}{2}\frac{P}{1+P}$

$$\Rightarrow \dot\rho_{33} = \Gamma\varepsilon(1-\rho_{33})\frac{P}{2(1+P)} \equiv A(1-\rho_{33})$$

$$\Rightarrow \rho_{33} = (1-e^{-At}). \Rightarrow t_{1/2} = \frac{2\ln 2\,(1+P)}{\varepsilon P}\Gamma^{-1}$$

4. $i\partial_t = -\frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x$

A. $\vec{v} = tr[\vec{\sigma}\rho] \Rightarrow v_x = tr\begin{bmatrix}\rho_{21} & \rho_{22}\\ \rho_{11} & \rho_{12}\end{bmatrix} = \rho_{21}+\rho_{12}$, $v_y = tr\begin{bmatrix} i\rho_{21} & i\rho_{22} \\ -i\rho_{11} & -i\rho_{12}\end{bmatrix} = i(\rho_{21}-\rho_{12})$, $v_z = \rho_{11}-\rho_{22}$

$$\vec{v}\cdot\vec{\sigma} = v_x\sigma_x + v_y\sigma_y + v_z\sigma_z = \begin{bmatrix}\rho_{11}-\rho_{22} & 2\rho_{12}\\ 2\rho_{21} & \rho_{22}-\rho_{11}\end{bmatrix} = 2\underbrace{\begin{bmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{bmatrix}}_{\rho_{ij}} - 1$$

$$\Rightarrow \rho = \frac{1}{2}(1+\vec{v}\cdot\vec{\sigma})$$

B. $i\partial_t\vec{\sigma} = [\vec{\sigma}, H] = -\frac{\Delta}{2}[\vec{\sigma},\sigma_z] + \frac{\Omega}{2}[\vec{\sigma}.\sigma_x]$

$$= -\Delta(-i\sigma_y, i\sigma_x, 0) + \Omega(0, -i\sigma_z, i\sigma_y)$$

$$\Rightarrow \partial_t\vec{\sigma} = (\Delta\sigma_y, -\Delta\sigma_x - \Omega\sigma_z, \Omega\sigma_y) = (\Omega,0,-\Delta)\times\vec{\sigma} \equiv \vec{\omega}\times\vec{v}$$

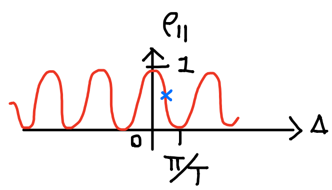
C. Step 1: $\vec{v} = (0.0,-1)$

Step 2: rotate $\pi/2$ along $(1.0.0)$ gives $\vec{v}=(0.1,0)$

Step 3: rotate along $(0,0,-\Delta)$ gives $\vec{v} = (\sin\Delta T, \cos\Delta T, 0)$

Step 4: rotate $\pi/2$ along $(1.0.0)$ gives $\vec{v} = (\sin\Delta T, 0, \cos\Delta T)$

Final $v_z = \cos\Delta T = \rho_{11}-\rho_{22} \Rightarrow \rho_{11} = \frac{1}{2}(1+\cos\Delta T)$



d. We may set $\Delta=\pi/2T$ and thus $\vec{v}=(1.0.0)$. Each atom will give $\rho_{11}=0$ or 1.

N atoms offer N indep. measurements. yielding an uncertainty of the mean estimator $\Sigma\rho_i/N$ of $\frac{1}{2\sqrt{N}}$. The system has a sensitivity to freq drift of $\delta\rho/\delta\Delta = \frac{T}{2} = \frac{1/2\sqrt{N}}{\delta\omega} \Rightarrow$ freq sensitivity is $\delta\omega = \frac{c}{T\sqrt{N}}$, where $c=1$.