2nd method

Rotating frame and Rotating Wave Approximation RWA

Idea: go to the frame that co-rotates with the laser.

$$i\partial_t \varphi = \frac{1}{2}\begin{pmatrix}\omega_0 & 0\\ 0 & -\omega_0\end{pmatrix}\varphi + \Omega\cos\omega t\begin{pmatrix}0&1\\1&0\end{pmatrix}\varphi$$

Coordinate transformation $\varphi = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2}\end{pmatrix}\phi \equiv \hat R\phi$

L.H.S. $= \begin{pmatrix}\omega/2 & 0\\ 0&-\omega/2\end{pmatrix}\hat R\phi + \hat R\, i\partial_t\phi$

R.H.S. $= \begin{pmatrix}\omega_0/2&0\\ 0&-\omega_0/2\end{pmatrix}\hat R\phi + \Omega\cos\omega t\begin{pmatrix}0&1\\1&0\end{pmatrix}R\phi$

Multiply both sides by $R^{-1} = \begin{pmatrix}e^{i\omega t/2}&0\\ 0&e^{-i\omega t/2}\end{pmatrix}$. we get

$$i\partial_t\phi = \frac{\omega_0-\omega}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\phi + \Omega\, \underline{\cos\omega t}\, \underline{R^{-1}\begin{pmatrix}0&1\\1&0\end{pmatrix}R}\,\phi$$

0

$(e^{i\omega t}+e^{-i\omega t}) * \begin{pmatrix}0 & e^{i\omega t}\\ e^{-i\omega t} & 0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix} + \cancel{\begin{pmatrix}0 & e^{2i\omega t}\\ e^{-2i\omega t} & 0\end{pmatrix}}$

This is the rotation wave approximation (RWA)

We get $i\partial_t\phi \approx -\frac{\Delta}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\phi + \frac{\Omega}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\phi$

No more time dependence!!

$$\partial_t\phi = \frac{-i}{2}\begin{pmatrix}\Delta & \Omega\\ \Omega & -\Delta\end{pmatrix}\phi$$

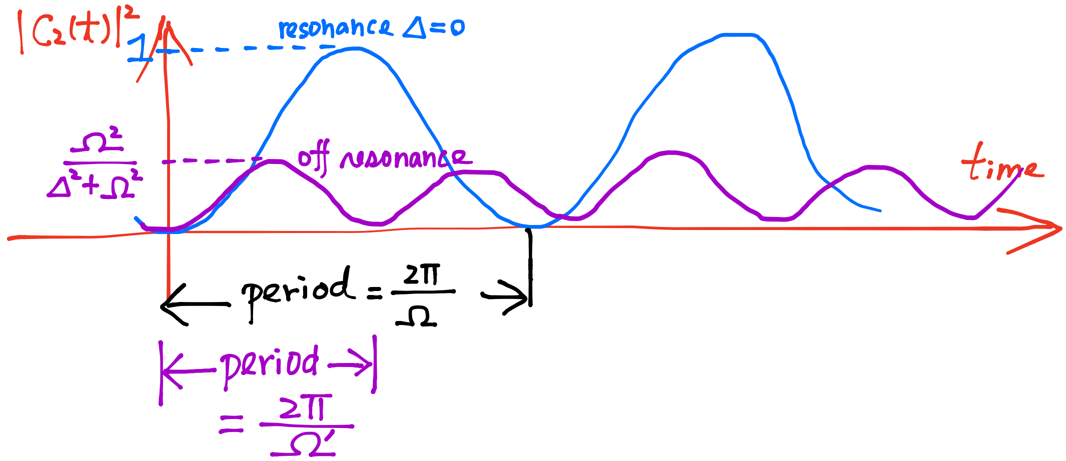
Assume $\phi = \begin{pmatrix}c_2\\c_1\end{pmatrix}$ and $c_1(0)=1$. $c_2(0)=0$, we get

$$|c_2(t)|^2 = \frac{\Omega^2}{\Delta^2+\Omega^2}\sin^2\sqrt{\Omega^2+\Delta^2}\,\frac{t}{2}$$

↑ generalized Rabi frequency

In general, $\phi(t) = \hat U(t)\phi(0)$. Derive the evolution operator $\hat U(t) = e^{-i\hat H t/\hbar}$.

Result:



On resonance $\Delta=0$. We can equally splite the amplitude at $t=\frac{\pi}{2}\Omega^{-1}$

This is the beam splitter we are looking for.

we have a $\pi/2$ pulse at $t=\frac{\pi}{2}\Omega^{-1}$.

Writing the general solution $\phi(t)=U(\Omega t)\phi(0)$, we define the evolution operator

$U(0)=\hat{1}$, $U(\pi/2)\sim\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$ ($\pi/2$ pulse): beam splitter

$U(\pi)\sim\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ (π pulse): population exchange

In the hw. you will derive the general evolution operator $U(t)$

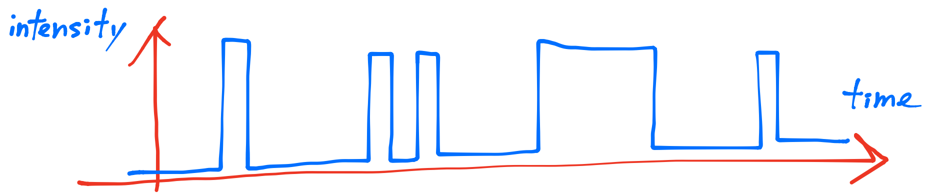
$\phi(t_2)=U(t_2-t_1)\phi(t_1)$ and show that

$U(\frac{\pi}{2}\Omega^{-1})$ is equivalent to a $\pi/2$ pulse

What are the explicite forms of $U(\pi/2)$, $U(\pi)$ and $U(2\pi)$?

10/9/2018 Bloch vector and Bloch sphere (Foot: 7.3)

In practice. many pluses are involved, we need to
Establish a simple physics intuition.



Build a physics picture

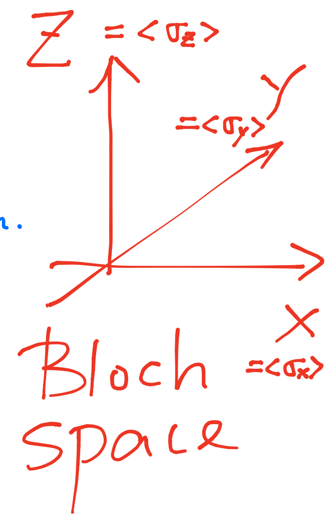
z $\begin{pmatrix}1\\0\end{pmatrix}$ ↑ $\begin{pmatrix}0\\1\end{pmatrix}$ ↓

x $\begin{pmatrix}1\\1\end{pmatrix}$ → $\begin{pmatrix}-1\\1\end{pmatrix}$ ←

y $\begin{pmatrix}i\\1\end{pmatrix}$ ↗ $\begin{pmatrix}-i\\1\end{pmatrix}$ ↙

How about $\begin{pmatrix}\pm 1\\1\end{pmatrix}$ and $\begin{pmatrix}\pm i\\1\end{pmatrix}$?

We need 2 more degrees of freedom.



Have we consider all possibities?

check dimensionality:

$\phi = \begin{pmatrix}C_2\\C_1\end{pmatrix}$ has 3 dim. (1 global phase) $\quad \vec{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ has 3-dim.

Spinor Wavefunction $\phi \Leftrightarrow$ Vector in 3D real space

Isomorphism (Group theory)

explain the approx. iso-morphism.

Operation of a spinor $\begin{pmatrix}C_2\\C_1\end{pmatrix} \approx$ Rotation of a 3D vector

SU(2)	SO(3)
Special unitary group (2x2)	Special orthogonal group (3x3)
All $\hat{U}$ that preserve probability	All rotatitions preserving length.

Formal definition of Bloch vector $\vec{b} = \langle \vec{\sigma} \rangle$ and $\vec{\sigma} = \langle \hat{\sigma}_x . \hat{\sigma}_y . \hat{\sigma}_z \rangle$

Pauli matrices are $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Check $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives $\vec{b} = (0.0.1)$ Note that global phase cancelled.

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ gives $\vec{b} = (1.0.0)$

$\frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$ gives $\vec{b} = (0.1.0.)$

Schrödinger Eqn for $\phi \Rightarrow$ Eqn. of motion for $\vec{b}$

Derivation I: Density matrix $\rho = |\rangle\langle|$ & Dipole operator (See Foot 7.3.2)

$$\rho = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (c_1^* \, c_2^*) = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_2 c_1^* & |c_2|^2 \end{pmatrix} \equiv \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$\vec{b} = (u.v.w)$ $\quad u = \rho_{12} + \rho_{21}$

$v = -i(\rho_{12} - \rho_{21})$

$w = \rho_{11} - \rho_{22}$

all 2x2 Hamiltonians can be assumed this form, why?

Derivation II: Heisenberg picture $i\hbar \partial_t \langle F \rangle = [F.H]$

Now $i\hbar \partial_t \vec{b} = \langle [\vec{\sigma}.H] \rangle$

We assume $H = \hbar(h_0 \hat{1} + h_1 \hat{\sigma}_x + h_2 \hat{\sigma}_y + h_3 \hat{\sigma}_z) \equiv \hbar \sum_{i=0}^{3} h_i \sigma_i$, $\quad \sigma_0 \equiv \hat{1}$

$$
\begin{aligned}
i \partial_t \vec{b} &= \sum_{i=0}^{3} h_i \langle [\vec{\sigma}.\sigma_i] \rangle \\
&= \sum_{i=1}^{3} h_i \langle [\vec{\sigma}.\sigma_i] \rangle \\
&= \sum_{ij}^{3} h_i \langle [\sigma_j.\sigma_i] \rangle \hat{e}_j \\
&= 2i \sum_{ijk} \langle \epsilon_{jik} h_i \sigma_k \hat{e}_j \rangle \\
&= 2i \sum_{ijk} \epsilon_{kij} h_i \langle \sigma_j \rangle \hat{e}_k \\
&= 2i \sum_{ijk} \epsilon_{ijk} h_i \langle \sigma_j \rangle \hat{e}_k
\end{aligned}
$$

$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$

$\epsilon_{ijk} = 1$ for $(1.2.3)$ and all even permutations

$= -1$ for odd permutations

$= 0$ for repeated index

$$\partial_t \vec{b} = 2\vec{h} \times \langle \vec{\sigma} \rangle = 2\vec{h} \times \vec{b}$$ $\Leftarrow$ Bloch vector eqn of motion.

Dynamics of Bloch vector $\approx$ angular rotation in mechanics

$\dot{\vec{b}} = 2\vec{h}\times\vec{b}$ $\qquad$ $\dot{\vec{r}} = \vec{\omega}\times\vec{r}$ or $\vec{\tau} = \dot{\vec{L}} = \vec{\mu}\times\vec{B} = \gamma\vec{B}\times\vec{L}$

gyromagnetic ratio

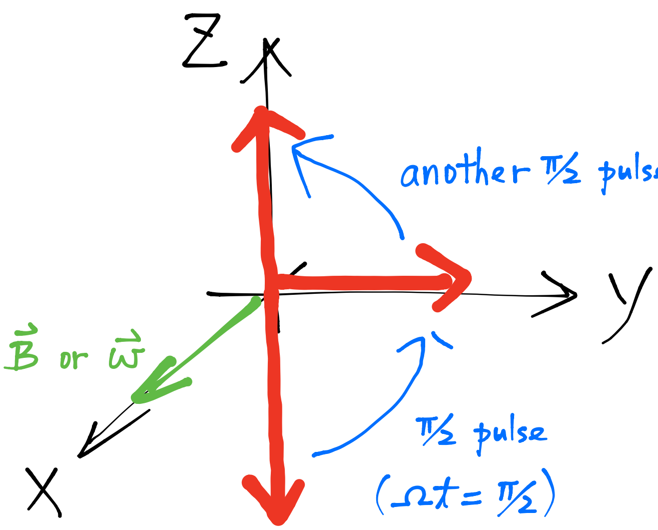
Back to our system $i\partial_t\phi = -\frac{\Delta}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\phi + \frac{\Omega}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\phi$

$$= \frac{1}{2}(\Omega, 0, -\Delta)\cdot(\sigma_x, \sigma_y, \sigma_z)\,\phi$$

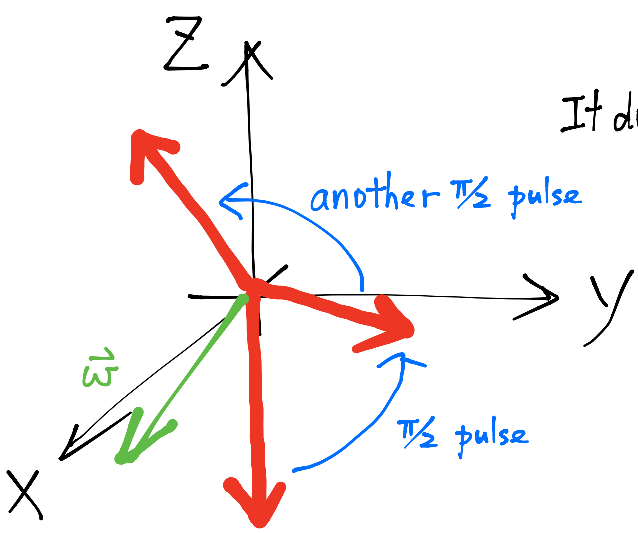
Bloch vector motion: $\partial_t\vec{b} = (\Omega, 0, -\Delta)\times\vec{b}$

On resonance $\Delta = 0$.

$\Rightarrow$ rotation along X axis



off-resonance excitation



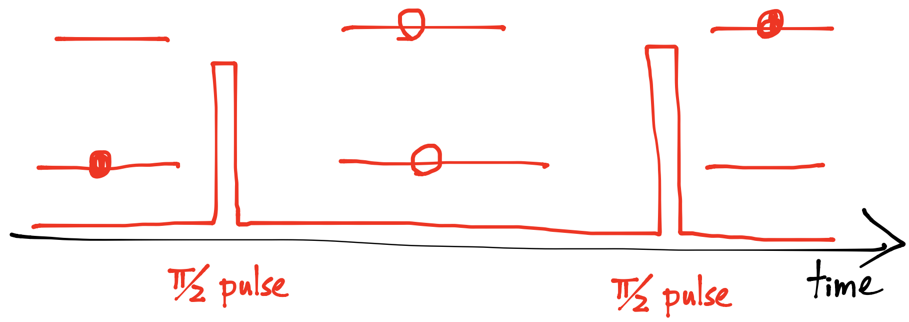
It doesn't reach $|C_2|^2 = 1$ anymore

but it rotates faster at

$$\Omega' = \sqrt{\Omega^2 + \Delta^2}$$

(generalized Rabi freq)

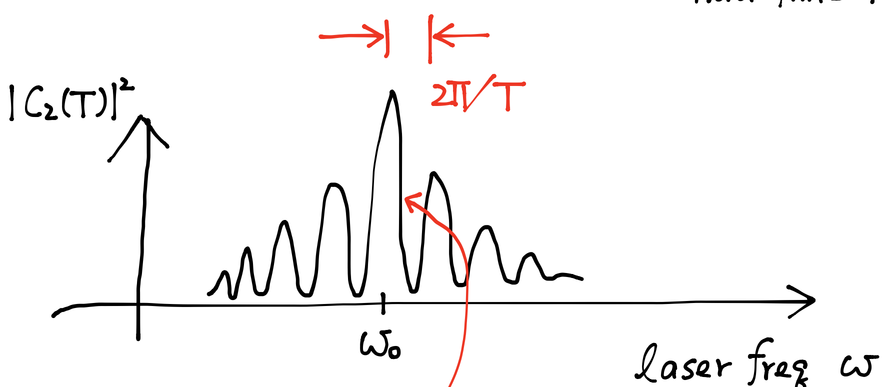
Back to "Ramsey Spectroscopy"



$\phi_i = \begin{pmatrix}0\\1\end{pmatrix}$ $U(\pi/2)$ free evolution $U(\pi/2)$ $\phi_f = ?$

Homework: show that $\phi_f = i\begin{pmatrix}-\cos\omega_0 T/2\\ \sin\omega_0 T/2\end{pmatrix}$

Thus



Again if we sit on the slope we can determine freq to an accuracy of $\delta\omega = \frac{\pi}{T}\frac{1}{SNR} = \frac{\pi}{T\sqrt{N}} \Rightarrow T\Delta E = \hbar\pi/\sqrt{N}$