Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2020

Problem Set #1

Due: 11:59 pm, Thursday, April 16. Please submit to Canvas.

1. **MATH** Please solve the following differential equations (10 points each)

   (a) $x'' + 3x' + 2x = 0$ with initial condition $x(0) = 1$ and $v(0) = 1$.
   (b) $x'' + 2x' + 2x = 0$ with initial condition $x(0) = x0$ and $v(0) = v0$.
   (c) $x'' + 2x' + 2x = \cos t$ with initial condition $x(0) = x0$ and $v(0) = v0$.
   Extra credits (5 points each)
   (d) $x' + x = \sin t$ with initial condition $x(0) = 1/2$.
   (e) Given the solution $x(t) = (1 + e^t)\cos t$, determine the differential equation that it satisfies with no explicit appearance of exponential factor e^t.

2. **Driven oscillator** (12 points each) You are testing a resonant circuit you find in the garage and wonder if it is any good. You hook it up to a signal generator, drive it at various frequencies, and measure its steady state response with a voltmeter. The data is summarized in the following table.

Driving freq. $\frac{\omega}{2\pi}(MHz)$	980	990	1000	1010	1020
Amplitude A (Volt)	32.04	71.33	70.80	31.52	19.46

   (a) Use the driving oscillator model we learned in the class and estimate the natural frequency $\frac{\omega_0}{2\pi}$ in Hz and the quality factor $Q = \frac{\omega_0}{\gamma}$.
   (Hint: The data are generated based on the model. Please determine values of $\frac{\omega_0}{2\pi}$ and Q to 3 digits.)
   (b) At what driving frequency (in terms of $\frac{\omega}{2\pi}$) would you expect the highest amplitude?
   (Hint: It will be close, but not exactly at the natural frequency due to the finite value of Q)

3. **Damped oscillator** (12 points each) One issue with the high-quality oscillator is that it damps slowly and will continue ringing even when you stop driving it. (Think about any music instrument.) One way to quickly damp out its motion is to connect it to a strong damper. It is an art to choose the right damper in order to dissipate the energy at the shortest amount of time. We assume the oscillator follow the model we learned in the class $x'' + \gamma x' + \omega_0^2 x = 0$.

   At t=0, we apply the damper which increases the damping coefficient to γ^*.

   (a) A standard scheme is to go to the critical damping ($\gamma^* = 2\omega_0$). Estimate the time the energy of the oscillator drops by a factor of e.

(b) Why not go to even larger damping, say, $\gamma^* = 20\,\omega_0$? Show that in the limit of very heavy damping $\gamma^* \gg \omega_0$, the kinetic energy of the oscillator $E_k = mx'^2/2$ decays quickly, but the potential energy $V = m\omega_0^2 x^2/2$ decays very slowly. More explicitly, Show that

$$E_k(t) = E_k(0)e^{-\alpha t}$$

$$V(t) = V(0)e^{-\beta t}$$

and calculate the decay rates α and β in this limit.

4. **Energy transfer** (12 points each) A high-Q oscillator can store an immense amount of energy. Let's consider the extreme case of a harmonic oscillator with negligible friction (thus $Q \to \infty$) and natural frequency ω_0.

(a) If we drive the oscillator with the force $f\cos\omega t$ at a frequency very close to the resonance $\omega = (1-\epsilon)\,\omega_0$ with $\epsilon \ll 1$. Show that the total energy of the oscillator $E = E_k + V$ after the steady state is reached is

$$E_{max} = \frac{mf^2}{8\omega_0^2\epsilon^2}.$$

(b) If the oscillator is stationary at $x(0) = x'(0) = 0$ and we apply the driving force $cos(\omega t)$ at t=0, show that the total energy increases from zero quadratically as

$$E(t) = \frac{1}{8}mf^2t^2$$

(c) Combine the above results, show that it takes a long time to charge a high-Q oscillator for small detuning $\epsilon \ll 1$

$$T \sim \frac{1}{\omega\epsilon}$$

(Hint: you may expand $\epsilon \ll 1$ to leading order.)