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Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2020

Problem Set #4

Due: 11:59 pm, Thursday, May 7. Please submit to Canvas.

1. **Dirac Delta function** *δ(x)* (11 points each)

   We may define the Delta function based on the following procedure

   - $f(x)$ is any function that has an integrated area of $\int f(x)dx = 1$.
   - Dirac's delta function is defined as $\delta(x) = \lim_{\Delta\to 0}\frac{1}{\Delta}f\left(\frac{x}{\Delta}\right)$

   (a) A common choice of f by physicists is the Gaussian function $f(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$. Use the form and prove that $\delta(x)$ satisfies the following properties

   1. $\delta(x \neq 0) = 0$
   2. $\delta(x = 0)$ diverges
   3. $\int \delta(x)dx = 1$
   4. $\int g(x)\delta(x-a)dx = g(a)$.

   (Hint: Apply the definition.)

   (b) Calculate or prove the following

   1. $\int g(x)\delta(ax+b)dx$
   2. $\int g(x)\frac{d\delta(x)}{dx}\,dx \qquad , g(\pm\infty) = 0$
   3. $\delta(x-\alpha) + \delta(x+\alpha\ ) = 2|\alpha|\delta(x^2-\alpha^2)$.

   (c) Solve the motion of a simple harmonic oscillator which is periodically kicked at period 1. Determine the condition that the harmonic oscillator becomes unstable.

   $$x'' + \omega_0^2 x = f(t) \equiv \sum_{n=\cdots,-1,0,\pm 1\ldots} \delta(t+n)$$

   (Hint: the standard approach is to start with Fourier expanding the external force. You may also use the fact that the integral $\int_{n-\epsilon}^{n+\epsilon}\delta(t-n)dt = 1$ for arbitrary small internal of 2ϵ.)

2. Energy and energy flow in a wave (11 points each)

   Here we will investigate how is energy distributed and propagating in a generic medium (string, air, water…). Assume the wave (transverse or longitudinal) satisfies the following wave equation

   $$\rho\partial_t^2\psi(x,t) = T\partial_x^2\psi(x,t),$$

   where ρ is the linear density of the medium and T is the "tension force" in the medium.

   (a) Given a small section between x and $x+\Delta x$, calculate its kinetic energy density $\partial_x E_k$ and potential energy density $\partial_x U$.

(Hint: First determine what the “potential energy” is for the section. Think about how the restoring force $-kx$ is linked to the potential energy $\frac{kx^2}{2}$ in a SHO.)

(b) Given a traveling wave $\psi(x,t) = A\cos k(x - vt)$, what is the total energy density $\partial_x E = \partial_x E_k + \partial_x U$. Explain why the total energy is time-dependent and is zero when the magnitude of the displacement $|\psi|$ reaches the maximum value(, where we expect to see large potential energy)?

(c) Given the traveling wave $\psi(x,t) = A\cos k(x - vt)$, determine the amount of energy that passes the position x per unit time in the propagation direction. This energy transfer (energy flux) can be written as $J = -T\partial_x\psi \cdot \partial_t\psi$.
Show that the above results suggest

$$J = v_E \partial_x E,$$

where the velocity of the energy flow is the same as that of the waveform $v_E = v = \sqrt{\frac{T}{\rho}}$.

3. Decibel scale of the strength of sound (11 points each)
Alexander Bell, the inventor of phone, introduced the unit of bel, which became decibel in acoustics: Zero decibel (0 dB) is defined as $\pm 20\ \mu Pa$ in the variation of air pressure ($\mu Pa = 10^{-6} Pa$), the typical threshold of human perception. Decibel is presented in log scale such that 20 dB corresponds to 200 μPa, 40dB 2 *mPa*, 60dB 20 *mPa* and so on.

(a) What is the assumed threshold of human perception? Calculate the intensity of a 1D acoustic wave at 0 dB.
(Hint: Intensity is energy delivered per area per time in the unit of Watt/m^2.)

(b) Our hearing is damaged above 100dB, calculate the intensity of the wave. Show that sound cannot be louder than 200dB.

(c) How much is the maximum displacement $\psi(x)$ of air molecules away from equilibrium in the presence of acoustic waves at 0dB and frequency = 100 Hz?

Material	Density (kg/m^3)	Compressibility (1/GPa)
Air	1.22	7200
Water	1000	0.5
Copper	8960	0.0073

(GPa = 10^9 Pa.)