Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2020

Problem Set #5

Due: 11:59 pm, Thursday, May 14. Please submit to Canvas.

1. (Math) Vector calculus (20 points each)
   a) A vector field $\vec{A}(x, y, z)$ carries a source or sink if $\nabla \cdot \vec{A} \neq 0$ and it carries circulation if $\nabla \times \vec{A} \neq 0$. Determine which function(s) carry no sources, no sinks and no circulation?
   $\vec{A} = (\mathrm{e}^{\mathrm{x}}, 1, \mathrm{x}\,\mathrm{e}^{\mathrm{y}})$
   $\vec{B} = (x^2 y, -\mathrm{x}\mathrm{y}^2, 0)$
   $\vec{C} = (\mathrm{yz}, \mathrm{zx}, \mathrm{xy})$
   $\vec{D} = (\cos z\,, \sin x - \sin z\,, -\cos x)$

   b) Prove the following vector identities (ϕ is a scalar field, $\vec{A}$ is a vector field.)
   $\nabla \cdot (\phi \vec{A}) = \phi \nabla \cdot \vec{A} + (\vec{A} \cdot \nabla)\phi$
   $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$
   $\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi$

2. Doppler effect and shock waves (20 points)
   Police uses Doppler radar gun to identify speeding violators. Assume the gun emits sound waves at frequency ω, and police detects the frequency of the reflected waves as ω_r.
   a) A car is moving straight away from the police at the speed v, show that the frequency of the reflected radio waves has a frequency of

$$\omega_r = \omega_0 \frac{v_p - v}{v_p + v}$$

   Hint: The radar gun emits waves with wavefronts forming concentric circles propagating outwards. The wavefronts are spaced by the wavelength λ and are moving outwards at the sound speed v_p. When the wavefronts hit the car, they are reflected back. The frequency of the reflected wave is given by the time duration the police receives two consecutive wavefronts.

   b) In the presence of strong wind, the above formula needs corrections. Assume the wind is in the same direction as the car and its speed is w, show that the reflected frequency is

$$\omega_\mathrm{r} = \omega_0 \frac{v_p - w}{v_p + w} \frac{v_p - v + w}{v_p + v - w}$$

   Hint: When the air moves at velocity w, sound propagates faster at speed $v_p + w$ if it goes in the same direction, and slower $v_p - w$ if it goes against the wind. The frequency of the echo here is again determined by the duration the police receives two consecutive wavefronts from the wave reflected by the car.

c) Mach number is the ratio of the vehicle speed to the sound speed $M = \frac{v}{v_p}$. For a supersonic jet, show that the Mach number can be determined from the shock angle θ as $sin\theta = \frac{1}{M}$. How fast is the jet moving in the photo with θ=58° and sound speed = 343 m/s?