Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2020

Problem Set #6

Due: 11:59 pm, Thursday, May 21. Please submit to Canvas.

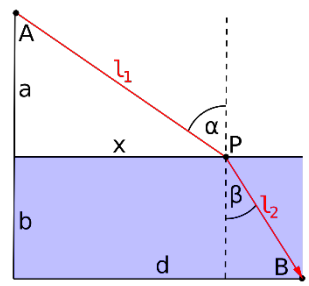
1. Fermat's principle (11 points each)

   We derived the Snell's law $n_1 \sin\theta_1 = n_2 \sin\theta_2$ based on Maxwell's equations, derived in 1861. The law, however, was discovered centuries ago by many, including Snellius (1580~1626) and P. Fermat in 1662 using the Fermat principle of least time:

   "The path taken by a ray between two given points is the path that can be traversed in the least time." *-Wikipedia*

   a) Given the air index of refraction $n=n_1$ and water $n=n_2$, a light beam passes through point A above the water and point B below the water. See figure for the dimensions. Show that when the travel time $T(x) = \frac{l_1(x)}{c/n_1} + \frac{l_2(x)}{c/n_2}$ is minimized, you get Snell's law $n_1 \sin\alpha = n_2 \sin\beta$.

   b) Does this mean the principle is always right?



   Hint: What if I stick a straw filled with air along the straight line from A to B, will the light "realize" it and quickly abandon the refractive path (red line)?

2. Electromagnetic (EM) waves in conductors. (12 points each)

   EM waves propagate differently in metal than in insulators since electrons can move and contribute to the fields. We will investigate how EM waves propagate in an ohmic conductor with charge density ρ and current density $j = \sigma E$, where σ is the conductivity. The Maxwell equations are

$$\nabla \cdot E = \frac{\rho}{\epsilon}$$
$$\nabla \cdot B = 0$$
$$\nabla \times E = -\partial_t B$$
$$\nabla \times \mathrm{B} = \mu\epsilon\, \partial_t E + \mu j$$

   a) Show that the electric field and magnetic field satisfy the wave equation:

$$\mu\epsilon\partial_t^2 E + \mu\sigma\partial_t E = \nabla^2 E - \frac{1}{\epsilon}\nabla\rho$$
$$\mu\epsilon\partial_t^2 B + \mu\sigma\partial_t B = \nabla^2 B$$

   b) Assume there is no free charge $\rho = 0$, show that the solution of an EM wave propagating in the z direction can be written as

   $E(z,t) = E_0 e^{-\frac{z}{z_0}} e^{ik(z-v_p t)}$,

   where the skin depth $z_0 \geq 0$ determines the decay of the field in the conductor, and v_p is the phase velocity. Show that they are

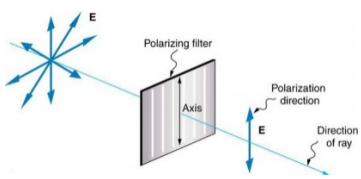
$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}}\right)^{-1/2}, z_0 = \frac{2}{v_p\mu\sigma}.$$

(Hint: Assume $E = e^{i(\tilde{k}z - \omega t)}$ and $\tilde{k} = k + i/z_0$, which can help simplify the calculation.)

c) The waves propagate slower in the conductor than in an insulator with the same μ and ϵ, where $v_p = \frac{1}{\sqrt{\mu\epsilon}} \equiv \frac{c}{n}$!! Show that the In a good conductor $\sigma \gg \omega\epsilon$, the phase propagation speed actually approaches zero like $v_p \approx \frac{c}{n}\sqrt{\frac{2\omega\epsilon}{\sigma}}$ and the group velocity is $v_g \approx 2v_p$. Based on the above criterion, at what frequency is aluminum ($\sigma = 3.8 \times \frac{10^7}{\Omega\, m}, \epsilon \approx \epsilon_0$) no longer considered as a good conductor?

d) Radiation intensity is given by the Poynting vector $< S(z,t) >= \frac{1}{\mu} < E(z,t) \times B(z,t) >_t = \frac{1}{2\mu v_p} E^2 = I_0 e^{-2z/z_0}$. Household microwaves operate at up to 1kW, $\frac{\omega}{2\pi}$ =2.4 GHz and about 40 cm in size. Such radiation is attenuated by its door. If the door is made of aluminum foil of 20 μm in thickness. Estimate (order of magnitude) the leakage power and compare that to sunlight 1,367 W/m^2, FCC microwave regulation limit of 160W/m^2.
(Hint: Standard microwave oven doors use thicker metal mesh with thickness > 0.7 mm.)

3. Polarizers (10 points each)
Polaroid polarizing filters are made of nitrocellulose polynemer film, where the polymers form needle-like crystals that only absorb light with polarization (E field) along the direction of the crystals. The axis of the polarizer is typically defined as the direction that the light can transmit. See figure, where the polarizer axis is in the x-direction, and the light propagates in z.



a) Assume the incident beam electric field (polarization) is given by $E_i = (E_x, E_y)$, and the polarizer is aligned in the x direction as shown in the figure. The transmitted beam is given by $E_t = E_x(1,0)$. A second polarizer rotated by $+\frac{\pi}{2}$ relative to the first one is placed after the first one. Show or argue that the transmission is zero.

b) Now we add a third polarizers between the two polarizers at an angle θ relative to the first one. Show that light can transmit again with electric field $E_i = E_x(0, \sin\theta\cos\theta)$.

c) Show that when an incident beam with electric field $E_{in} = (E_x^{in}, E_y^{in})$ passes through a polarizer rotated by $+\theta$ relative to the x-axis, the outgoing field is given by

$$E_{out}^T = \begin{pmatrix} E_x^{out} \\ E_y^{out} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} E_{in}^T.$$

(Hint: E^T means the transpose of vector $E = (E_x, E_y)$, use superposition principle.)