Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2020
Problem Set #7
Due: 11:59 pm, Thursday, May 28. Please submit to Canvas.

1. Brewster angle and total internal reflection (8 points each)
   Consider an EM wave in medium 1 with index of refraction n_1 entering medium 2 with index of refraction n_2. The incident angle θ_1, the same as the reflection angle, and the refraction angle θ_2 satisfy Snell's law. Based on the derivation in the class show that
   a) 100% refraction occurs at the Brewster's angle.
      Show that for p-polarized waves, there is a particular incident angle $\theta_1 = \theta_B$ that the reflection vanishes and 100% of the light transmits. Determine θ_B and show that the reflected and refracted waves are perpendicular to each other $\theta_1 + \theta_2 = \pi/2$.
   b) You can explain the use of polarized sunglasses to remove glare from the sunlight reflecting off horizontal surface such as water or ground.
   c) 100% reflection also occurs when $n_2 < n_1$ and the incident angle satisfies $\sin \theta_1 > \frac{n_2}{n_1}$.
      This phenomenon, called total internal reflection, finds enormous applications in optical communication. Show that when the incident angle is exactly at the critical value $\theta_c = \arcsin \frac{n_2}{n_1}$, there is a weird strong wave propagating exactly along the interface. Determine its amplitude in terms of E^t/E^i for s-polarized light.
   d) Beyond the critical incident angle $\theta > \theta_c$, "refracted wave" still exists and propagates in the second medium with decaying, non-oscillating amplitude. This decaying, non-oscillating wave is called the "evanescent wave". Estimate the decay length for the s-polarized light

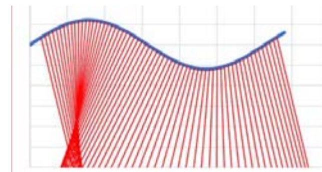
      Hint: Since Snell's law does not apply, we need to go back to the boundary condition $E(z = 0^+) + E^r(z = 0^+) = E^t(z = 0^-)$, see Lecture 11-3. Show that in the x-direction the wavenumbers are $k \sin \theta_1 = k_x^t$, assuming the transmitted wavenumber is $\vec{k}^t = (k_x^t, k_z^t)$. Then apply $k^t = n_2\omega/c$ and $k^t = n_2\omega/c$, show that $k_z^t = i/\lambda$ must be imaginary. The wavefunction is thus $\psi^t \propto \exp\left(-\frac{z}{\lambda}\right)$.

2. Protect the Great Lakes from invasive fish (8 points each)
   Army Corps signed off a $778 million plan to keep Asian carp (grass carp) from invading the Great Lakes last summer (2019). The dominant path of the invasion is along the Chicago Sanitary Canal (30 miles in length, 50 m wide and 2.7 m in depth). Let's brainstorm an idea to keep the fish away. If it works, Army may fund our proposal.

   Different from other fish the carp stay, live and breed at the bottom of the river. So how about let's focus sunlight to the bottom of the canal by generating a standing water wave on the surface?

   a) Assume the depth of the river is $D(x) = D_0 + A \sin kx$, where D_0 is the nominal depth, A<D is the water wave amplitude, x is the distance along the lake, and the index of refraction of the air is n_1 and water n_2. At what depth is the sunlight focused?

(Hint: The water wave surface can act as a lens to focus or defocus the light. See figure. Consider the wave function at the location that has the strongest focusing power and use the formula we derived based on Fermat's principle and paraxial approximation to determine the focal length.)



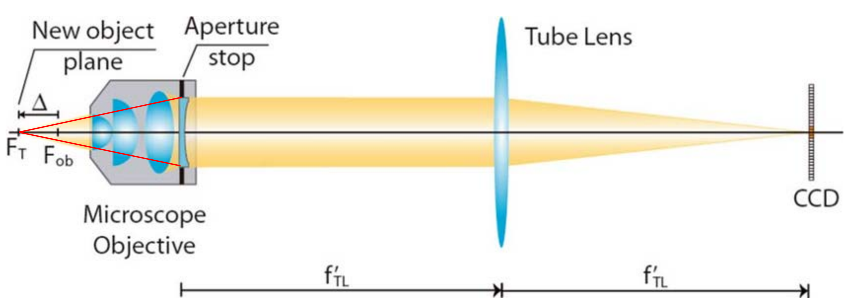
(Hint: Use $\frac{n_1}{P}+\frac{n_2}{Q}=2(n_2-n_1)\frac{z}{x^2}$ (see Lecture 12-1) and determine the focus when the sunlight goes straight down to the surface.)

b) Given the typical wavelength on the river of about $\lambda=\frac{2\pi}{k}=1.5$m, $n_1=1$, and n_2=4/3, what amplitude is needed to focus the light to the fish?

c) If the sunlight intensity is I_0, estimate how much high is the averaged intensity on the fish of size 30cm? Neglecting reflections from the surface and absorption in the water.

(Hint: Use Snell's law and estimate how far the sunlight should deviate from the peak so the ray will miss the fish. You may perform leading order expansion beyond the paraxial approximation or numerically find the solution.)

3. Optical microscope. (8 points each)
We will illustrate the difference between geometrical and Fourier optics with an example: microscope. A simple model for the microscope is



We consider an object on the left side of the objective with the working distance of *x*. (Working distance is defined as the distance between the object and the closest surface of the objective.) For simplicity, we approximate the objective as a thin lens, and it collimates the light from the object onto the tube lens. The tube lens then forms an image on the CCD. Assume the objective has a focal length of *f* and the tube lens has a focal length of *F*.

a) Draw the ray diagram for a point light source located at *x=f* away from the objective and show that in the above design, the distance between the objective and the tube lens does not matter (called infinite conjugation) and the CCD should be placed away from the tube lens by exactly F.

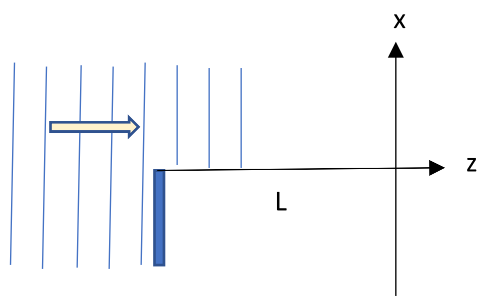
b) Show that the image on the CCD is upside down and magnified by a factor of *M=F/f*.

(Hint: Draw at least two rays from an off-axis point. Collimated ray should pass through the focus of the lens and vice versa. Ray that goes through the lens center will not be deflected.)

c) Use the lens equation and assume the light source drifts by $\delta x \ll x = f$, show that we need to move the CCD by $-M^2\delta x$ to regain a clear image. How much is the magnification changed?

Hint: When applying the lens equation $\frac{1}{D_0} + \frac{1}{D_i} = \frac{1}{f}$, note the convention that the object is assumed to be on the left side if $D_0 > 0$. The image is assumed to be on the right side if $D_i > 0$. We need a sign flip when either object or image is on the other side.
Hint: In multiple lens system the image of the one lens is the object of the next lens.

4. Fraunhofer diffraction by a sharp edge (8 points each)
One intriguing optical phenomenon is the diffraction from the sharp edge of an object that blocks the light. Assuming a plane wave propagating in the z direction is blocked by a thin barrier such that I(x<0, z=0)=0. The screen is L away from the edge.



To show the diffraction pattern in the far field we need to go beyond geometrical optics. Start with Huygens principle and assume paraxial approximation $x, y \ll L$, we derived

$$\psi(x, y, L) = e^{ikL} \int\int \psi(x', y', z = 0) A(x, y, z = 0) e^{ikx'x/L} e^{iky'y/L} dx'dy'$$

in Lecture 13-4 to calculate the single slit diffraction pattern with the aperture function $A\left(-\frac{d}{2} < x < \frac{d}{2}, y, 0\right) = 1$ and $A\left(|x| > \frac{d}{2}, y, 0\right) = 0$, and $\psi(x', y', z = 0)$ is the incident wave function.

The result is just the Fourier transform of $\psi(x', y')A(x, y)$ at $z = 0$.

~~a) Drop the y coordinate and determine the edge diffraction pattern $|\psi(x, z = L)|^2$ of a uniform incident beam $\psi(x', y', z = 0) = \sqrt{I_0}$ and the lower half is blocked by the aperture function $A(x > 0, z = 0) = 1$ and $A(x < 0, z = 0) = 0$.~~

b) Calculate the single slit diffraction pattern using the above method and the aperture function and show that the result that is consistent with our derivation in Lecture 13-4.

c) Derive the double slit interference pattern when two slits have a finite width *w* and their centers are separated by d > w.
(Hint: you may use $A(|x \pm d/2| > w/2, z = 0) = 1$ and $A(|x \pm d/2| < w/2, z = 0) = 0$.)

d) Sketch the intensity pattern of b) for *d=2w* and identify the location of the nodes.