Lecture 1: harmonic oscillator. Morin chapter 1.1 and 1.2

Lecturer:

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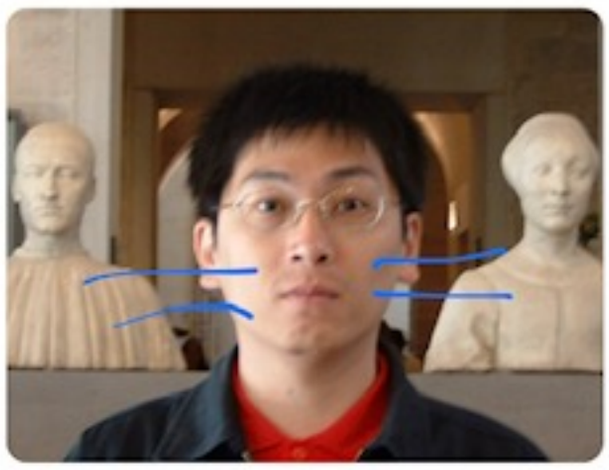
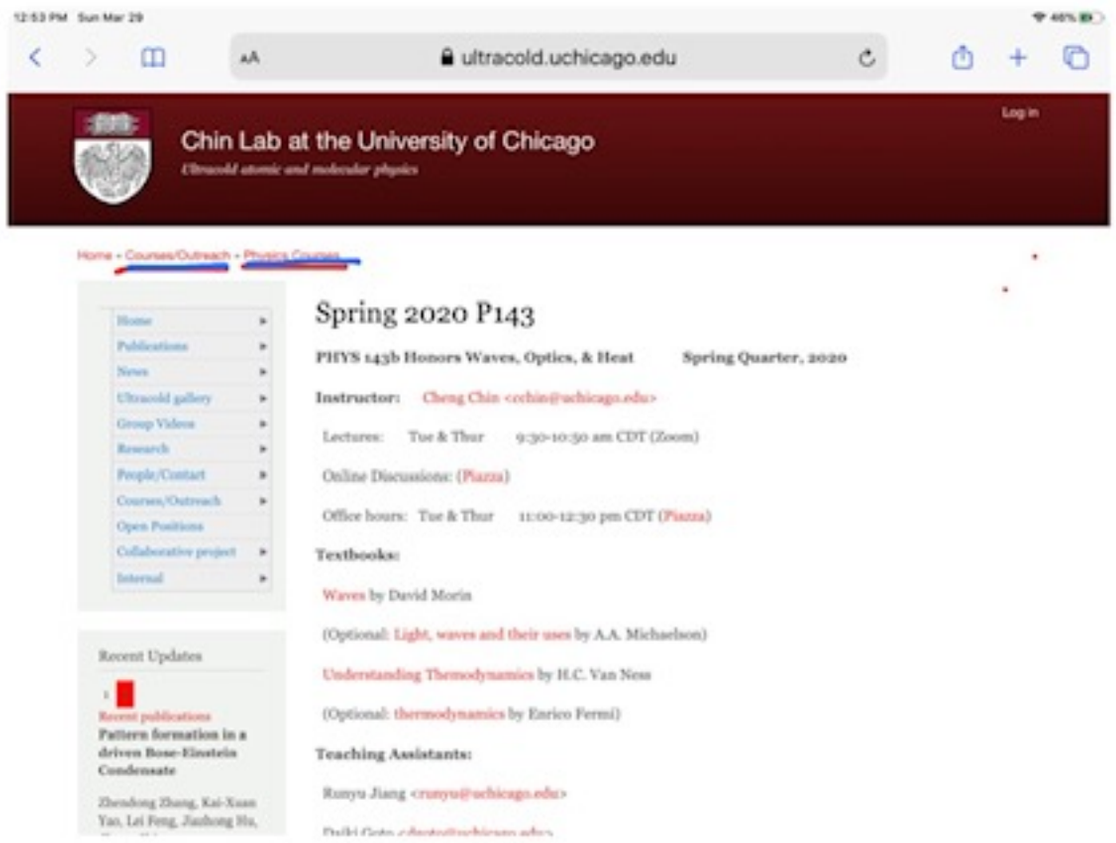


About the class:

- Syllabus: ultracold.uchicago.edu
- Asynchronous teaching: videos
- Labs: Phys. Dept Staffs
- Synchronous teaching: Zoom (143) →
- Discussion and office hours: piazza
- Grades distribution

Class website:
ultracold.uchicago.edu/node/176



Homeworks:

- Released on website.
- Due in one week
- Submit HWs to Canvas. (Also lab reports & Midterm)
- Late policy: −10% per day.

Midterm: 24 hours take home. submit to ~~TAs~~ Canv.

Final: not yet decided

Questions? Class, Lectures. discussion → Email Cheng
HWs, Midterm and final → TAs
Lab and reports → Lab staffs and TAs.

P143b Lecture 1-2 Solving ordinary differential equations

The eqn that concerns us many times in this class will be of the following form:

$$A\,x''(t) + B\,x'(t) + C\,x(t) = D\,f(t)\,,$$ A, B, C, D are constants

also written as

$$A\ddot{x}(t) + B\dot{x}(t) + C\,x(t) = D f(t) \quad \text{or}$$

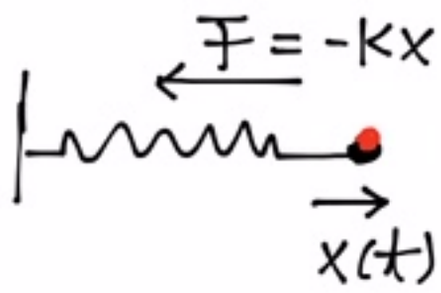
$$A\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + C\,x(t) = D f(t)$$

This is called 2nd order ordinary differential equation (ODE)

$\boxed{\text{only up to 2nd derivatives}}$ $\boxed{\text{one indep. variable } t}$

Example 1: $F = ma = m\,x''(t) \Rightarrow x''(t) = \frac{F}{m}$ is a 2nd order ODE

Example 2: point mass m attached to a spring with restoring force



$$\Rightarrow x''(t) = -\frac{K}{m}x(t)$$

$$\Rightarrow x'' + \frac{K}{m}x = 0$$

This is a "homogeneous" 2nd order ODE

no f(t) on the right side

Last definition: A differential eqn is called linear if ....

$x_1(t)$ is a solution, $x_2(t)$ is also a solution

$\Rightarrow$ $X = \alpha x_1(t) + \beta x_2(t)$ is also a solution

We will deal with linear, homogeneous ODE in this class.

Given $\left.\begin{array}{l} A x_1''(t) + B x_1'(t) + C x_1(t) = 0 \\ A x_2''(t) + B x_2'(t) + C x_2(t) = 0 \end{array}\right\}$ show that $A X''(t) + B X'(t) + C X(t) = 0$

$X \equiv \alpha x_1(t) + \beta x_2(t)$

You can see this by substitution!!

First case: Simple Harmonic oscillator

Potential

$U = \frac{1}{2}kx^2$

$\Rightarrow F = -\frac{dU}{dx} = -kx$

$\Rightarrow m x''(t) = F = -kx$

$\Rightarrow x''(t) + \frac{k}{m} x(t) = 0 \Rightarrow x(t) = ?$

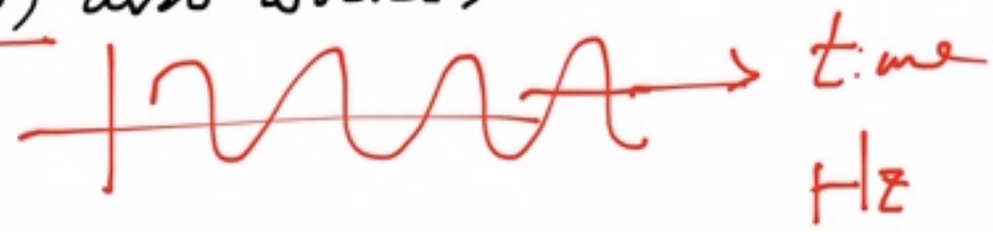
A good tip: Try sin and cos (Since we know the motion is periodic.)

Plug in $x = A\cos(\omega t + \phi)$ Trial func.

$\Rightarrow x'' = -A\omega^2 \cos(\omega t + \phi)$

$\Rightarrow \omega = \sqrt{k/m}$, A and ϕ are not constrained

Similarly, $A\sin(\omega t + \phi)$ also works)



A: amplitude

ω: angular freq (1/s) $\Rightarrow T = 2\pi/\omega$: period, $f = \omega/2\pi$: frequency (Hertz)

ϕ: phase

$= 1/T$

Since eqn is linear we know $A\sin\omega t + B\cos\omega t$ is also a solution. The values of A and B can be determined by the initial condition.

velocity $v = x'(t)$

If initial position $x(0) = 0$, velocity $x'(0) = v_0$

$\Rightarrow x(t) = A\sin\omega t$, $A\omega = v_0$

$v = 0$

If $x(0) = x_0$. $x'(0) = 0$

$\Rightarrow x(t) = x_0 \cos\omega t$

How about the general case $Ax''+Bx'+Cx=0$?

A great tip: Try $x(t)=e^{at}$

$\Rightarrow x'(t)=ae^{at}$

$\Rightarrow x''(t)=a^2e^{at}$

(Why? b/c exp function looks the same after differentiation.)

Plug in, we get $Aa^2\cancel{e^{at}}+Ba\cancel{e^{at}}+C\cancel{e^{at}}=0$

Since $e^{at}\neq 0$, we get $\boxed{Aa^2+Ba+C=0}$

We reduce ODE into an algebraic Eqn !!

We have 2 roots: $a=\frac{1}{2A}\left[-B\pm\sqrt{B^2-4AC}\right]\equiv a_1, a_2$

So the general solution is $x(t)=\alpha e^{a_1t}+\beta e^{a_2t}$

Again α and β can be determined by the initial conditions.

Let's try this tip on harmonic oscillator: $x''(t)+\frac{k}{m}x(t)=0$

Given $x(t)=e^{at}$, we get $a^2\cancel{e^{at}}+\frac{k}{m}\cancel{e^{at}}=0$

$\Rightarrow a^2=-\frac{k}{m} \Rightarrow a=\pm\sqrt{\frac{k}{m}}\,i\equiv\pm\omega i$ $\qquad i^2=-1$

$\Rightarrow$ General solution is $x(t)=Ae^{i\omega t}+Be^{-i\omega t}$

A and B are arbitrary constants

This is identical to $\boxed{x(t)=\bar{A}\cos\omega t+\bar{B}\sin\omega t}$, $\bar{A}$ & $\bar{B}$ are constants, since

$\boxed{e^{i\omega t}=\cos\omega t+i\sin\omega t}$

$\boxed{e^{-i\omega t}=\cos\omega t-i\sin\omega t.}$

equivalently

$\boxed{\cos\omega t=\frac{e^{i\omega t}+e^{-i\omega t}}{2}}$

$\boxed{\sin\omega t=\frac{e^{i\omega t}-e^{-i\omega t}}{2i}}$

Notes on complex numbers $Z = a + bi$, $i^2 = -1$, a, b are real.

$\boxed{a \equiv \mathrm{Re}[Z]}$ is the real part

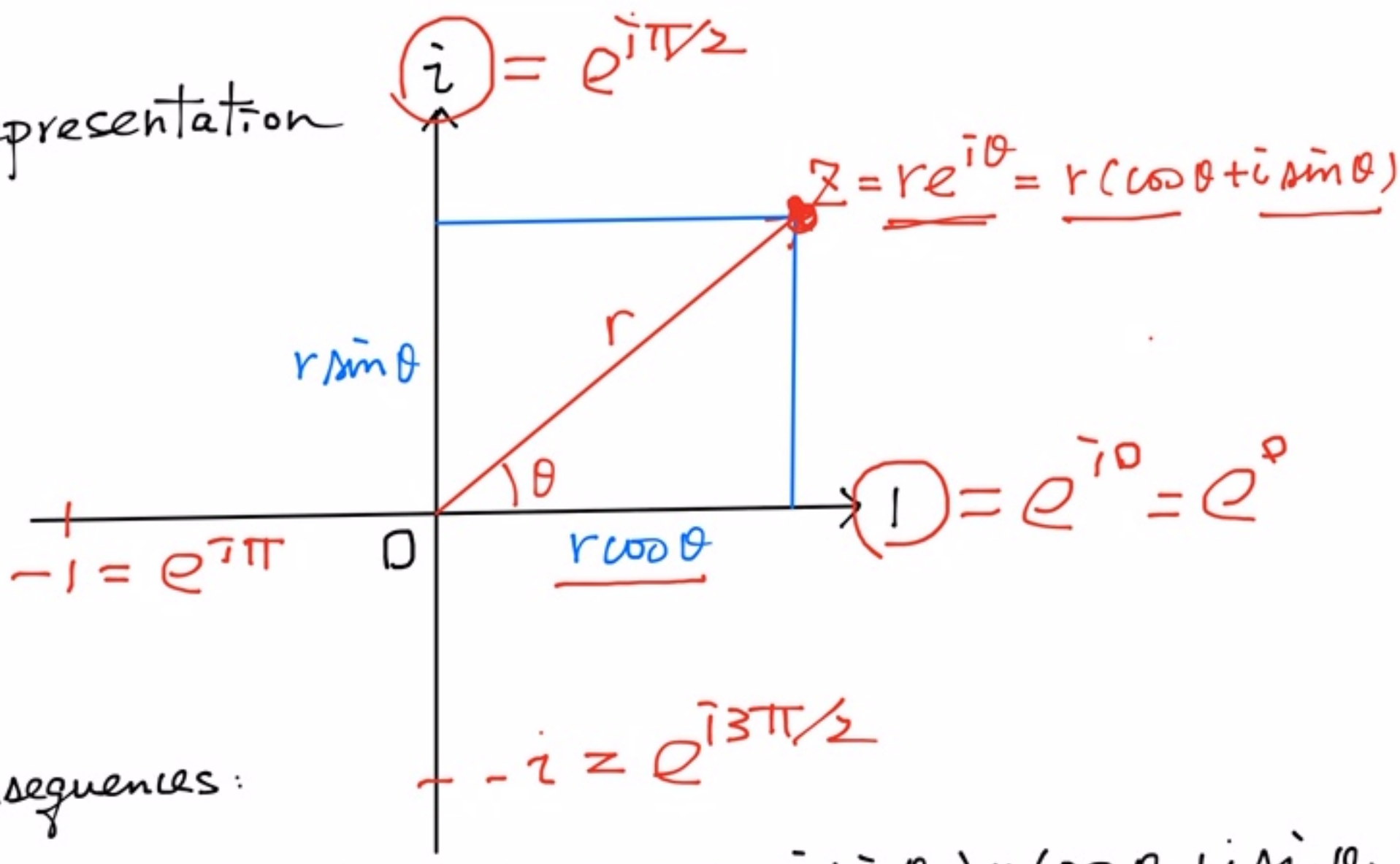
$\boxed{b \equiv \mathrm{Im}[Z]}$ is the imaginary part.

Exponential representation $Z = re^{i\theta} = \boxed{r\cos\theta + i\, r\sin\theta}$

$r \equiv |Z|$ is the absolute value

$\theta \equiv \mathrm{Arg}(Z)$ is the argument

Graphic representation



Very cool consequences:

(1) $e^{i\theta_1} * e^{i\theta_2} = ?$

$Z_1 * Z_2$

Stupid method: $(\cos\theta_1 + i\sin\theta_1) * (\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2$

$= \ldots\ldots$ Slow!!

Smart method: $e^{i\theta_1 + i\theta_2} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

(2) $$\frac{Z_2}{Z_1} = \frac{r_2 e^{i\theta_2}}{r_1 e^{i\theta_1}} = \frac{r_2}{r_1} e^{i\theta_2} e^{-i\theta_1} = \frac{r_2}{r_1} e^{i(\theta_2 - \theta_1)}$$

Fast!!

(3) $1 = e^0,\ i = e^{i\pi/2},\ -1 = e^{i\pi},\ -i = e^{-i\pi/2}$

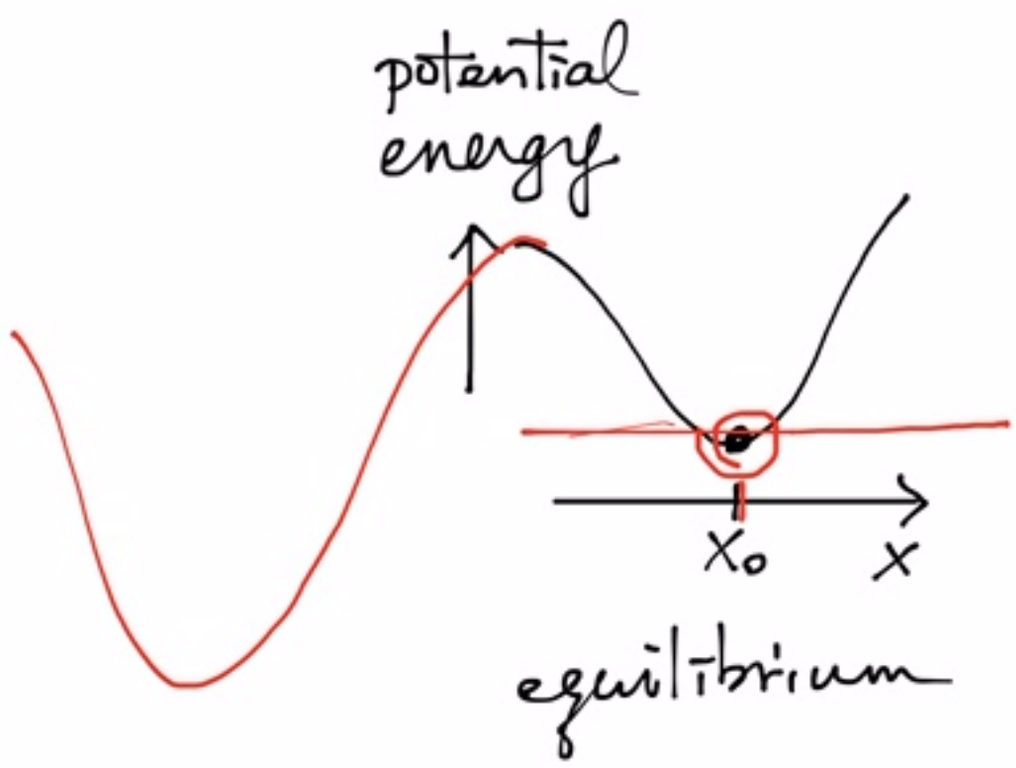
Lecture 1-3 Simple Harmonic Oscillator Cheng Chin

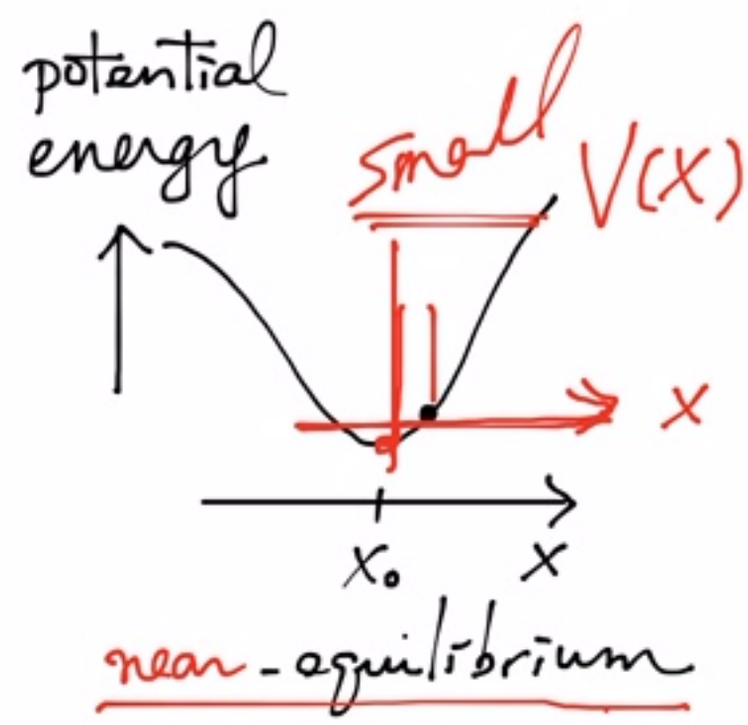
What are waves? Google it.

Water waves, sound waves, light waves. seismic waves, gravitational waves, shock waves. brain waves, matter waves, hand waves

- Waves are modulation of physical quantities in space & time.
- It occurs frequently when the system is driven slightly away from "equilibrium".

Simple harmonic oscillator (SHO) model





We can expand a generic potential energy around it's local minimum by Taylor expansion

$$V(x) = V(x_0) + V'(x_0)*(x-x_0) + \frac{1}{2}V''(x_0)*(x-x_0)^2 + \frac{1}{6}V'''(x_0)(x-x_0)^3 + \cdots$$

constant unimportant; $=0$ since slope $=0$ at the local minium; leading order non-zero term; higher order terms

If we only perturb the system gently, we ignore higher orders & get

$V(x) \approx \cancel{V(x_0)} + \frac{1}{2} V''(x_0)(x-x_0)^2$. If we further shift the origin to the equilibrium position $(x_0, V(x_0))$, we get

Harmonic potential $\quad V(x) = \frac{1}{2} k x^2, \quad k = V''(x_0)$

force constant

small deviation from equilibrium

Restoring force $F = -dV/dx = -kx$ (Hooke's law)

Motion of particle near the bottom:

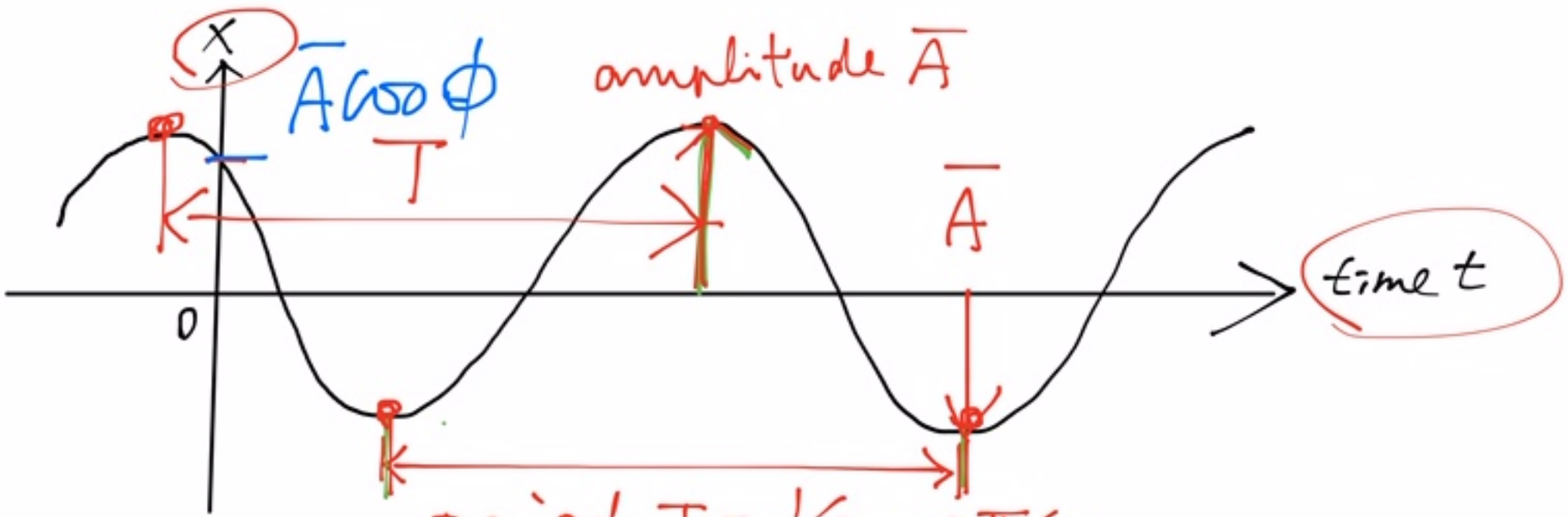
$$F = ma = \boxed{m x''(t) = -k x(t)}$$

$$\Rightarrow x(t) = A\cos\omega t + B\sin\omega t, \quad \omega = \sqrt{k/m}$$

see Lecture 1-2

or equivalently $x(t) = \bar{A}\cos(\omega t + \phi)$, or $x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$

Note: There are 2 independent constants determined by the initial conditions. Two because this is a 2nd order differential Eqn.



period $T = 1/f = 2\pi/\omega$

f: frequency in Hz

ω: angular freq. in 1/s

Given two initial conditions, we can determine A and B (or $\bar{A}$ and ϕ, C_1 and C_2)

$$x = A\cos\omega t + B\sin\omega t$$

Say we know initial position $x(0) = x_0$, velocity $x'(0) = v_0$

$$\Rightarrow \quad x(0) = A\cos 0 + B\cancel{\sin 0} = \boxed{A = x_0}$$

$$v(t) = x'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$v(0) = -A\cancel{\omega\sin 0} + B\omega\cos 0 = B\omega = v_0$$

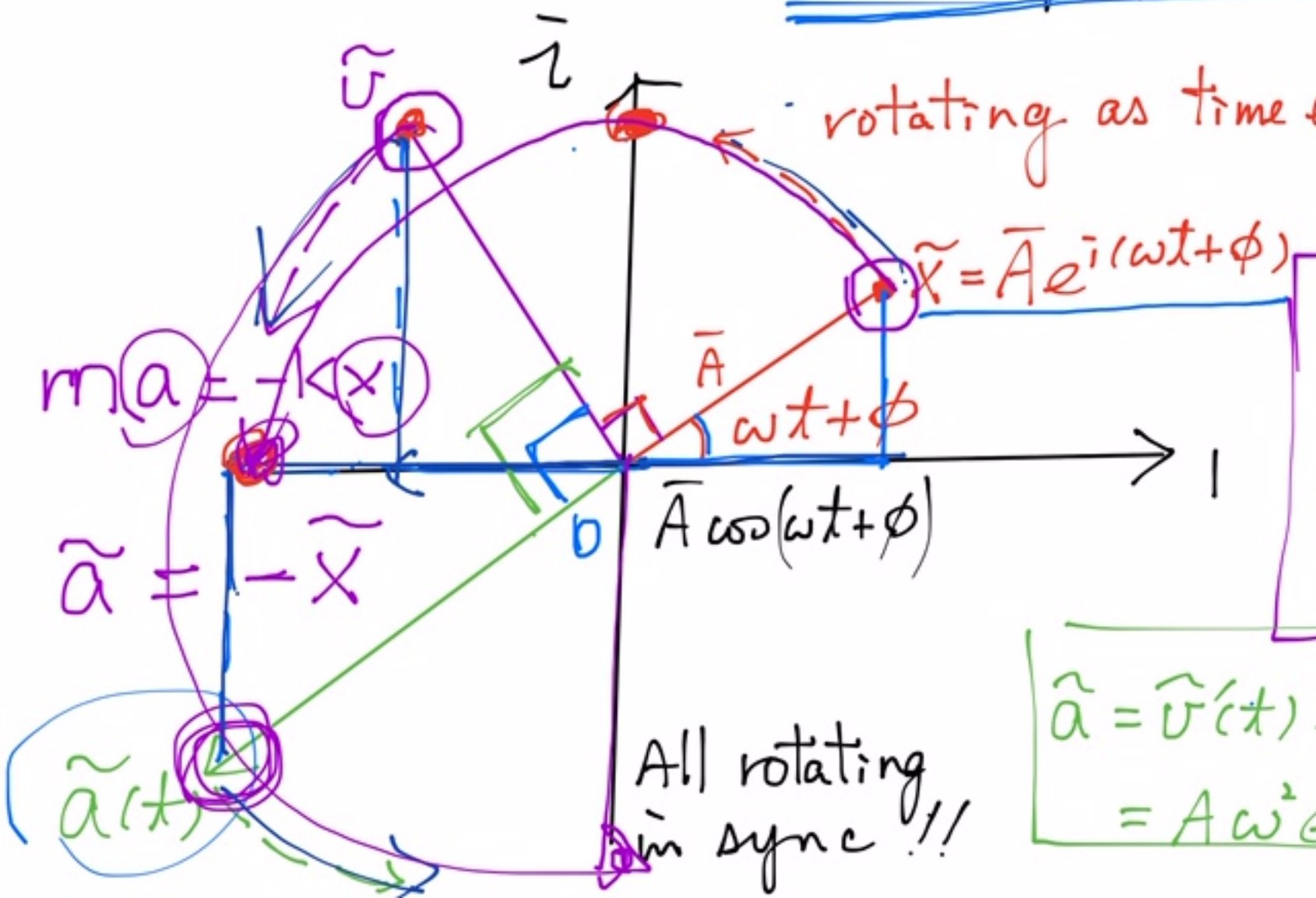
$$\Rightarrow \boxed{A = x_0,\ B = v_0/\omega}$$

* Here $\boxed{x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t}$ is a particularly simple form when we know x_0 & v_0.

If we know the amplitude and phase, you would prefer $x(t) = A\cos(\omega t + \phi)$. In other cases, the complex form is better.

Phasor diagram $\boxed{x(t) = \bar{A}\cos(\omega t + \phi)} = \mathrm{Re}\left[\underbrace{A e^{i(\omega t+\phi)}}_{\tilde{x}}\right]$

$= x$-axis projection of $\bar{A}e^{i(\omega t+\phi)}$



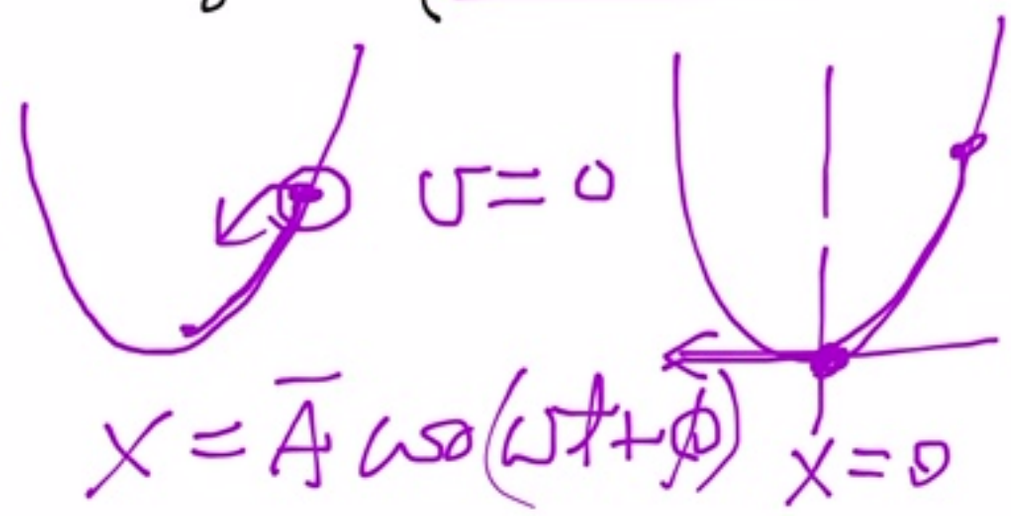
90°

$i = e^{i\pi/2}$

$$\tilde{v} = \tilde{x}'(t) = \bar{A}i\omega e^{i(\omega t+\phi)} = \boxed{A\omega}e^{i(\omega t+\phi+\frac{\pi}{2})}$$

$$\tilde{a} = \tilde{v}'(t) = -\omega^2 A e^{i(\omega t+\phi)} = A\omega^2 e^{i(\omega t+\pi)}$$

We can see when x reaches maximum $x=\bar{A}$, velocity is 0. When Velocity reaches maximum, displacement is zero! (Same applies to velocity and acceleration.)



Energy of Simple harmonic oscillator

potential energy $V = \frac{1}{2}kx^2 = \frac{1}{2}k\bar{A}^2\cos^2(\omega t+\phi)$

kinetic energy $E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2\bar{A}^2\sin^2(\omega t+\phi)$

Note that $\omega=\sqrt{k/m}$, so we have total energy

$$E = V+E_k = \frac{1}{2}k\bar{A}^2\cos^2(\omega t+\phi)+\frac{1}{2}k\bar{A}^2\sin^2(\omega t+\phi)$$

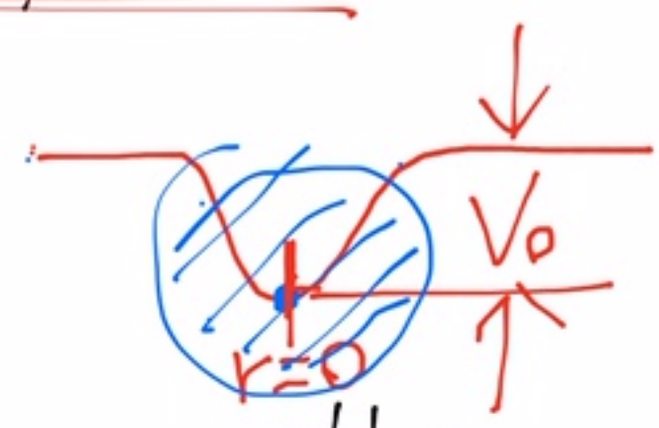
$$= \frac{1}{2}k\bar{A}^2 = \text{const. of time}$$

(Energy conservation)

Example: In our lab cesium atoms are trapped in a laser beam the generates a Gaussian potential

$$\boxed{V = -V_0 e^{-2r^2/r_0^2}}$$

trap depth; $r^2 = x^2+y^2$; beam waist



Calculate the vibration period near equilibrium.

$$F = -dV(r)/dr = \boxed{-V_0\, 2r/r_0^2\, e^{-2r^2/r_0^2}} \approx -\frac{2V_0}{r_0^2}r \quad -kr$$

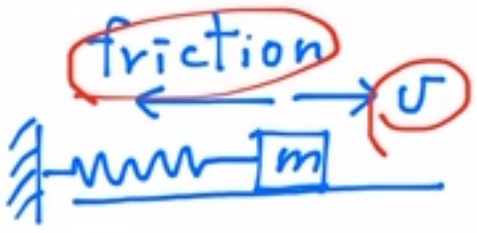
$$\Rightarrow mr'' + \frac{2V_0}{r_0^2}r = 0, \quad \omega = \sqrt{2V_0/mr_0^2} = \frac{1}{r_0}\sqrt{2V_0/m}$$

$$\Rightarrow \text{period} = 2\pi/\omega = \boxed{2\pi r_0\sqrt{m/V_0}}$$ ⇒ Short period in small beam.

Lecture 1-4 Harmonic oscillator with damping

Cheng Chin

First a demonstration that most oscillators damp out because of dissipation.



true for low velocity

friction force $F = -bv = -bx'(t)$

$$F = mx'' = -kx - bx' \Rightarrow \boxed{x'' + \gamma x' + \omega_0^2 x = 0}$$

We define $\gamma \equiv b/m$: friction coeff.

$\omega_0 = \sqrt{k/m}$: natural freq.

$\Rightarrow$ So this lecture we will discuss

$$x''(t) + \gamma x'(t) + \omega_0^2 x(t) = 0$$

Use the trip in Lecture 1-2, $x(t) = e^{\alpha t}$ $x'(t) = \alpha e^{\alpha t}$. $x'' = \alpha^2 e^{\alpha t}$

We get $\alpha^2 + \gamma\alpha + \omega_0^2 = 0$ Quadratic eqn.

$$\Rightarrow \alpha = \frac{1}{2}\left[-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right] \equiv \alpha_1, \alpha_2$$

$\Rightarrow x(t) = A e^{\alpha_1 t} + B e^{\alpha_2 t}$, where A and B can be determined by 2 initial conditions.

So what's new here?

discriminant $\gamma^2 - 4\omega_0^2 < 0$ small damping

- α_1 and α_2 can be complex $\gamma^2 < 4\omega_0^2$ (underdamping)
- α_1 and α_2 can be real $\gamma^2 > 4\omega_0^2$ (overdamping)
- α_1 and α_2 can be identical $\gamma^2 = 4\omega_0^2$ (critical damping)
- α_1 and α_2 can be purely imaginary $\gamma = 0 \Rightarrow$ SHO. $A\cos\omega_0 t + B\sin\omega_0 t$

We will discuss the first 3 cases here:

※ $\gamma > 0$. $\omega_0 > 0$, we do not consider unphysical case $\gamma < 0$

Underdamped oscillator ($\gamma < 2\omega_0$) $\Rightarrow \sqrt{\gamma^2 - 4\omega_0^2} = \boxed{\sqrt{4\omega_0^2 - \gamma^2}}\, i$ ← $\mathbb{R}$

means small

$$\Rightarrow \alpha = \frac{1}{2}\left(-\gamma \pm \sqrt{4\omega_0^2 - \gamma^2}\, i\right)$$

$$= -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \gamma^2/4} \equiv u \quad = -\frac{\gamma}{2} \pm iu \equiv \alpha_1, \alpha_2$$

$$\Rightarrow x(t) = C_1 e^{-\gamma t/2} e^{iut} + C_2 e^{-\gamma t/2} e^{-iut}$$

(1) $$= \boxed{e^{-\gamma t/2}}\left(C_1 e^{iut} + C_2 e^{-iut}\right)$$

This part is the same as we saw in SHO.

equivalently

(2) $$= e^{-\gamma t/2}\left(A\cos ut + B\sin ut\right)$$

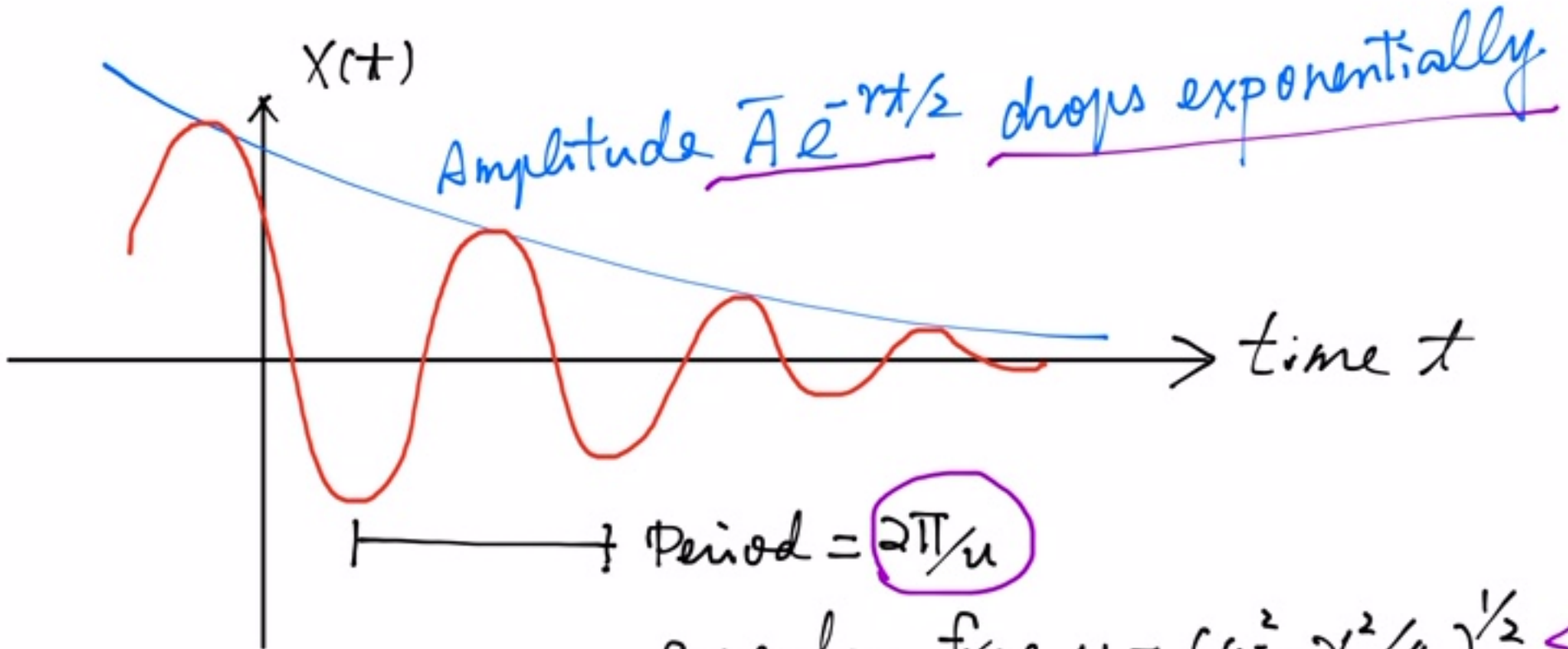
(3) $$= \boxed{e^{-\gamma t/2}\, \bar{A}\cos(ut + \phi)}$$

(4) $$= \mathrm{Re}\left[e^{-\gamma t/2} e^{i(ut+\phi)}\right]$$ (Phasor notation)

All of them have 2 degrees of freedom.

Choose the form that is easiest for your calculation

How does it look?



angular freq. $u = (\omega_0^2 - \gamma^2/4)^{1/2} < \omega_0$

Summary: Amplitude $= \bar{A} e^{-\gamma t/2}$ ← decay at rate $\gamma/2$

frequency shifts down to $(\omega_0^2 - \gamma^2/4)^{1/2}$

Total energy $E = \frac{1}{2} k x^2 + \frac{1}{2} m x'^2(t)$

show that $E(t) = E(0) e^{-\gamma t} \left[1 + \frac{\gamma^2}{4\omega_0^2} \cos(2ut + \phi)\right]$

Averaging over half cycle we get $\langle E \rangle = E(0) e^{-\gamma t}$

$\boxed{\Rightarrow \text{Energy damps out at the rate of } \gamma.}$

How many oscillations do we see before amplitude drops by e?

Define Quality value a.k.a. Q factor $Q = \omega_0/\gamma$ (dimensionless)

100

$A(t) = A(0) e^{-\gamma t/2} = A(0) e^{-1} \Rightarrow t = 2/\gamma$

We know period $= 2\pi/u \approx 2\pi/\omega_0$

$\times 3.14$

There are $\frac{2/\gamma}{2\pi/\omega_0} = \boxed{\frac{Q}{\pi}}$ oscillations

$\Rightarrow$ So if you see 100 oscillations $\Rightarrow Q \approx 300$, $\gamma = \frac{\omega_0}{300}$ good oscillator!!

if you see 1 oscillation $\Rightarrow Q = 3.14$ Barely an oscillator

Super-duper oscillators: Crystal oscillator $Q = 10^4 \sim 10^6$

Super-conducting resonators: $Q = 10^6 \sim 10^8$

Cs atomic clock: $Q = 10^{10}$

$\boxed{Q \gg 1}$

For $\boxed{Q = \frac{\omega_0}{\gamma} \gg 1}$, oscillation freq $u = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \approx \omega_0 \left(1 - \frac{\gamma^2}{8\omega_0^2}\right) = \omega_0 \left(1 - \frac{1}{8Q^2}\right)$

When $Q = 100$, frequency shifts by $\frac{1}{8Q^2} = 10^{-5} = 0.001\%$

Overdamped oscillator $(\gamma > 2\omega_0)$ So $Q = \frac{\omega_0}{\gamma} < 1/2$!!

bad ~~low~~ quality!!

Here the oscillators are so damped that it doesn't even oscillate:

$$\alpha = \frac{1}{2}\left[-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right]$$

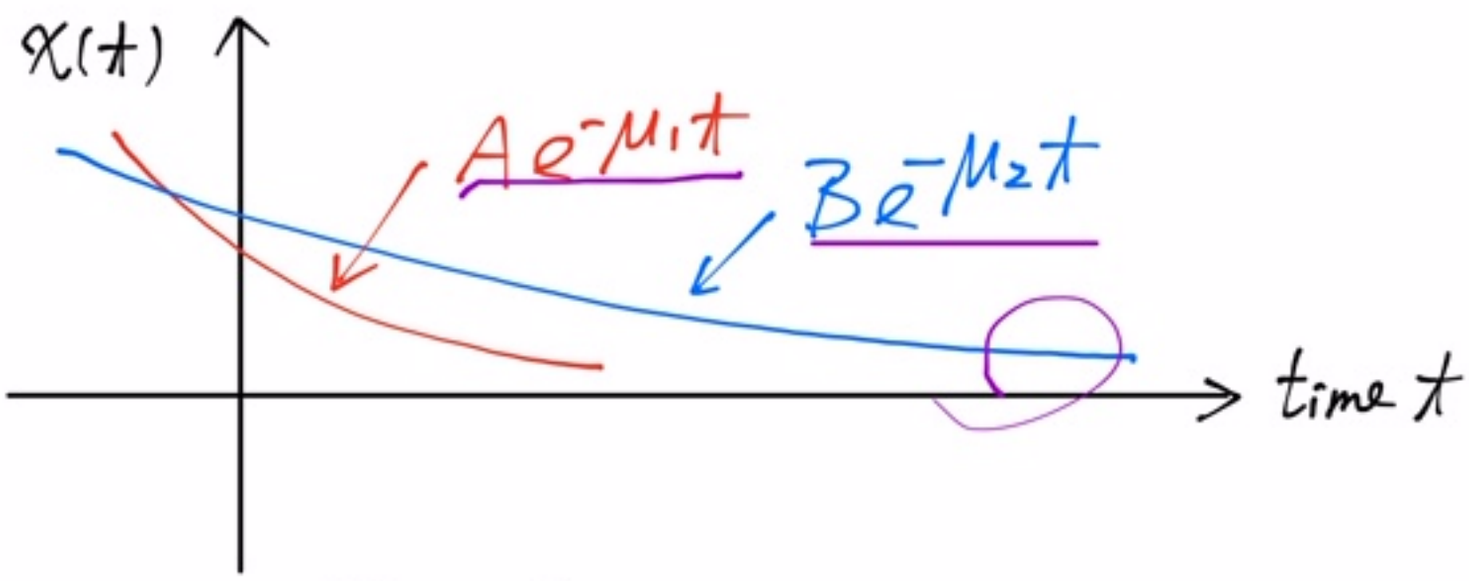
$$\equiv \alpha_1 \text{ and } \alpha_2$$

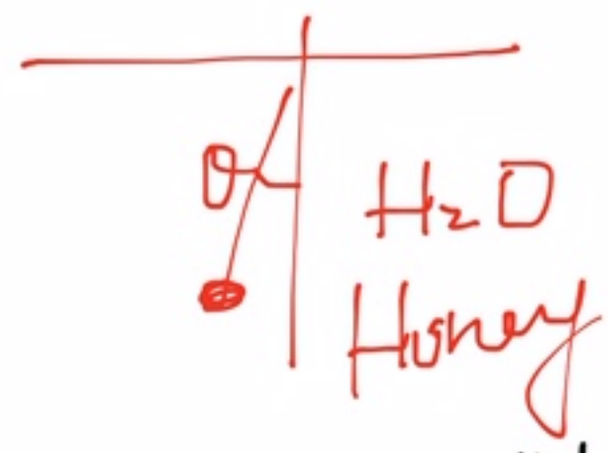
both roots are real and negative

$$\alpha_1 = -\frac{\gamma}{2}\left[1 + \sqrt{1 - 4\omega_0^2/\gamma^2}\right] \equiv -\mu_1$$

$$\alpha_2 = -\frac{\gamma}{2}\left[1 - \sqrt{1 - 4\omega_0^2/\gamma^2}\right] \equiv -\mu_2$$

$$x(t) = A e^{-\mu_1 t} + B e^{-\mu_2 t} \qquad \mu_1 > \mu_2 > 0$$





- No more oscillation !!
- There are two ways to damp out the motion, one fast $e^{-\mu_1 t}$, one slow $e^{-\mu_2 t}$

Why are there 2 ways to damp out the motion?

Can we gain some intuition from the strong damping limit $\boxed{\epsilon = \frac{\omega_0^2}{\gamma^2} \ll 1}$

$$\mu_1 = \frac{\gamma}{2}\left(1 + \sqrt{1 - 4\epsilon}\right) \approx \gamma \qquad \text{damping} = \gamma \quad \text{makes sense.}$$

$$\mu_2 = \frac{\gamma}{2}\left(1 - \sqrt{1 - 4\epsilon}\right) \approx \frac{\gamma}{2}\left[1 - (1 - 2\epsilon)\right] = \epsilon\gamma = \frac{\omega_0^2}{\gamma} \propto \frac{1}{\gamma}$$

With strong damping, there are 2 modes of motion:
One damps out fast when you increase ~~damping~~ γ
One damps out slow when you increase ~~damping~~ γ.
If we wait for long time slow motion dominates.

Wierd! What's going on??

Critical damping : $(\gamma = 2\omega_0)$

$$\alpha = \frac{1}{2}\left[-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right] = -\frac{\gamma}{2}, \text{ double root !!}$$

$$x = Ae^{-\gamma t/2} = Ae^{-\omega_0 t}$$

problem!! Well we have only one degree of freedom, how do we specify initial position & velocity independently?

$t^3 e^{\alpha t}$

$t^2 e^{\alpha t}$

How do we handle $x''(t) + \overset{\gamma}{2\omega_0} x'(t) + \omega_0^2 x(t) = 0$?

Math: when this happens, a special solution is $x(t) \propto \boxed{t e^{\alpha t}}$

Prove that $x(t) = te^{\alpha t}$ is indeed a solution and $\alpha = -\omega_0$

$\Rightarrow$ general solution is $x(t) = \boxed{(A + Bt)}\boxed{e^{-\omega_0 t}}$

Physics: What's going on, how can a new solution suddenly appear? New type of dynamics?

What happens is that $x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$ naturally approaches $x(t) = \bar{A}e^{\alpha t} + \bar{B}te^{\alpha t}$ when $\alpha_2 \to \alpha_1$

Proof: $Ae^{\alpha t} + Be^{(\alpha+\epsilon)t}$

$$= e^{\alpha t}(A + Be^{\epsilon t}) \xrightarrow[\epsilon \to 0]{} e^{\alpha t}[A + B(1 - \epsilon t)]$$

$$= e^{\alpha t}[A + B - B\epsilon t]$$

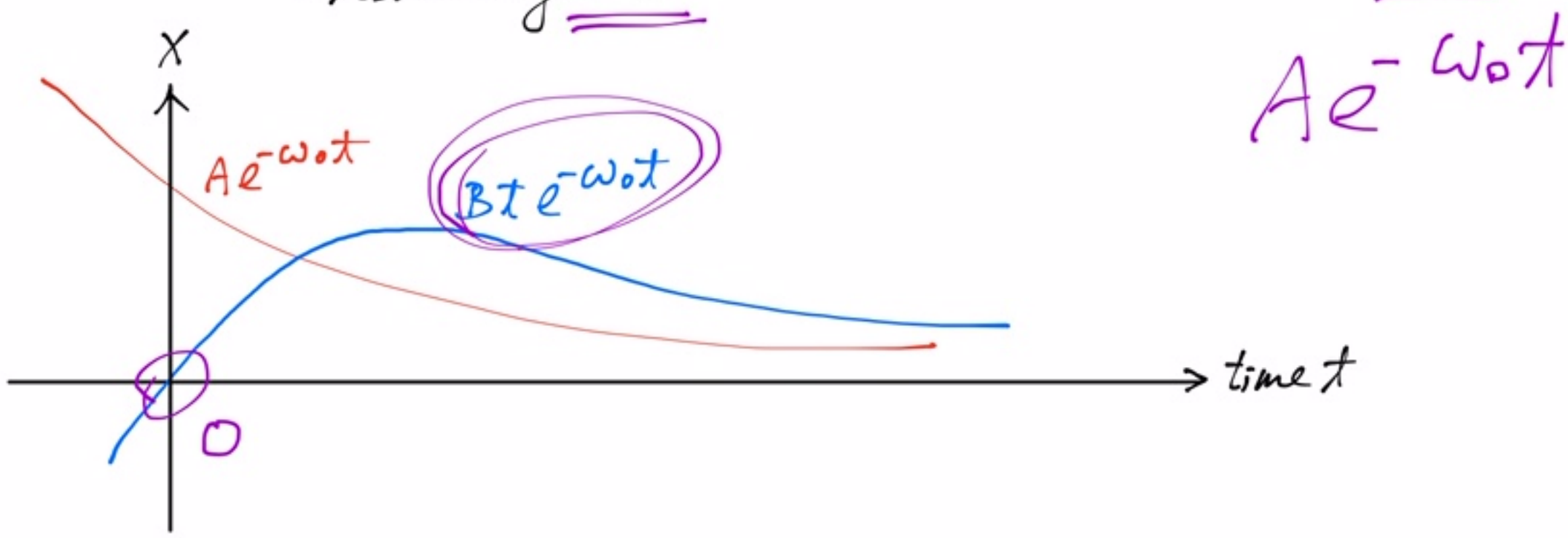
$$= (\bar{A} + \bar{B}t)e^{\alpha t}$$

$$\bar{A} = A + B, \quad \bar{B} = -B\epsilon$$

The solution is $x(t) = \boxed{(A+Bt)}\,\boxed{e^{-\omega_0 t}}$

* An interesting observation is that this solution passes zero once at $\boxed{t = -A/B}$

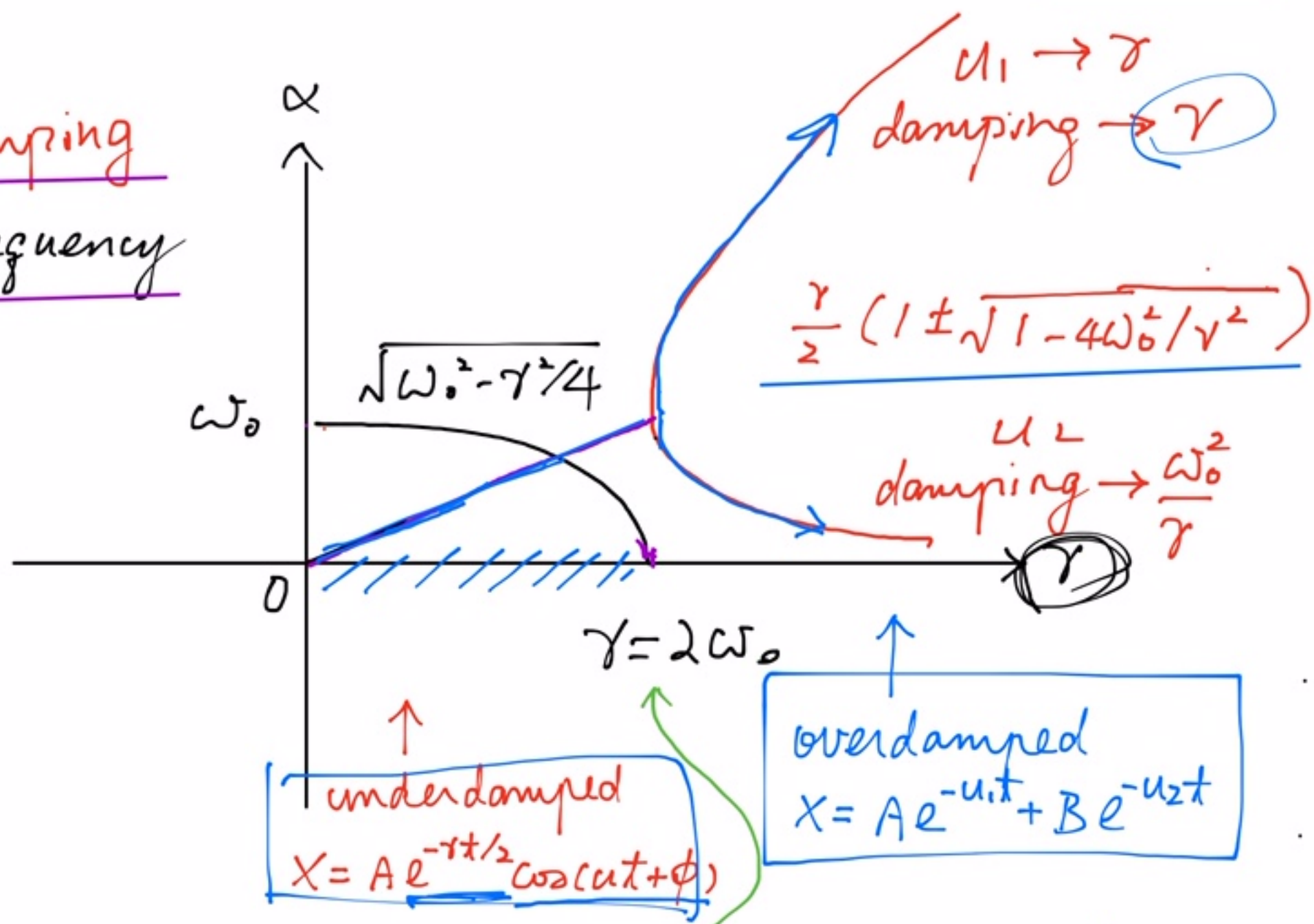
(Assuming $B \neq 0$. it would not pass $x=0$ if $B=0$)



Summary: Damped oscillator $x'' + \gamma x' + \omega_0^2 x = 0 \Rightarrow x \sim e^{\alpha t}$

$-\mathrm{Re}[\alpha] > 0$: damping

$\mathrm{Im}[\alpha] > 0$: frequency



critically damped: $x = (A+Bt)e^{-\omega_0 t}$

Energy dissipates to zero the fastest