Lecture 2-1 Solving Non-homogeneous ODE

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Definitions $x_H''(t) + \gamma x_H'(t) + \omega_0^2 x_H(t) = 0$ homogeneous ODE

$+)\ x_S''(t) + \gamma x_S'(t) + \omega_0^2 x_S(t) = f(t)$ non-homogeneous ODE

1 One of the most important properties is that $x_S + x_H$ is also a solution of the non-homo. ODE

To prove it, just add the 2 equations and get

$$(x_S + x_H)'' + \gamma (x_S + x_H)' + \omega_0^2 (x_S + x_H) = f(t)$$

Since we also know the homogeneous solution $x_H = A e^{\alpha_1 t} + B e^{\alpha_2 t}$

$\Rightarrow$ general form of the non-homo. ODE is $x_S + x_H = x_S + A e^{\alpha_1 t} + B e^{\alpha_2 t}$

Since A & B take up degrees of freedom of $x(0)$ & $x'(t)$, the special solution x_S has no more degree of freedom. So it is special.

2. If x_{S_1} is the special solution for $f_1(t)$, $x_{S_2}(t)$ is the one for $f_2(t)$, then $x_{S_1} + x_{S_2}$ is the solution for $f_1(t) + f_2(t)$

This is because the eqn is linear (prove it)

Say we want to solve $\boxed{x'' + \gamma x' + \omega_0^2 x = A\cos\omega t}$ (x, A real)

we can supplement it with $\boxed{y'' + \gamma y' + \omega_0^2 y = A\sin\omega t}$ $\times i$

Add the two eqns as $(x+iy)'' + \gamma(x+iy)' + \omega_0^2(x+iy) = A(\cos\omega t + i\sin\omega t)$

So we can instead solve $\boxed{\tilde{x}'' + \gamma \tilde{x}' + \omega_0^2 \tilde{x} = A e^{i\omega t}}$

and the real part of $\tilde{x}$ is the solution $x = \mathrm{Re}[\tilde{x}]$ that we want.

Similarly, $\mathrm{Im}[\tilde{x}]$ is the solution with $f(t) = A\sin\omega t$.

$\Rightarrow$ With complex #, we can solve 2 eqns at once
But the real reason is that calculation is simple.

Let's solve $x'' + \gamma x' + \omega_0^2 x = f\cos\omega t$ (driven damped harmonic oscillator)
driving force

by solving $\tilde{x}'' + \gamma \tilde{x}' + \omega_0^2 \tilde{x} = f e^{i\omega t}$ (in general $\omega \neq \omega_0$)

Tip for special solution: $\tilde{x} = A e^{i\omega t}$ ($\tilde{x}, A$ are complex)

$$\tilde{x}' = A i\omega e^{i\omega t}$$

$$\tilde{x}'' = -A\omega^2 e^{i\omega t}$$

Plug in everything: $-A\omega^2 \cancel{e^{i\omega t}} + A i\omega\gamma \cancel{e^{i\omega t}} + A\omega_0^2 \cancel{e^{i\omega t}} = f \cancel{e^{i\omega t}}$

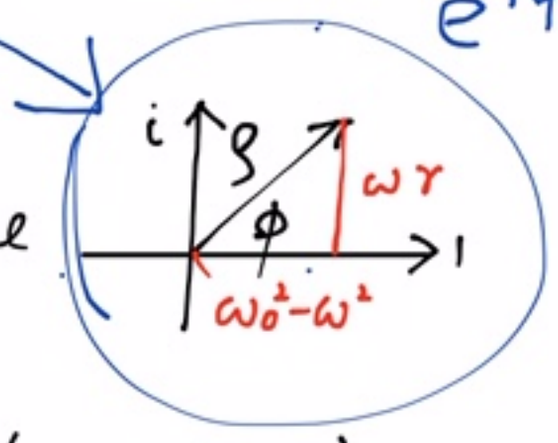
Since $e^{i\omega t} \neq 0 \Rightarrow A[-\omega^2 + i\omega\gamma + \omega_0^2] = f$

$$\Rightarrow \tilde{x} = \boxed{\frac{f}{\omega_0^2 - \omega^2 + i\omega\gamma} e^{i\omega t}}$$

$$\frac{1}{e^{i\phi}} = e^{-i\phi}$$

The real part of it is the desired solution.

$$\boxed{x_s = \mathrm{Re}[\tilde{x}] = \frac{f}{\underbrace{\sqrt{(\omega_0^2-\omega)^2 + \omega^2\gamma^2}}_{\equiv \rho}} \cos(\omega t - \phi)}, \text{ where}$$



(Note that there is no more degrees of freedom, A & ϕ are given)

Conclusion of the Math part:

Given $x'' + \gamma x' + \omega_0^2 x = f\cos\omega t$ the solution is

$$x(t) = x_s(t) + x_H(t)$$

$$= \frac{f}{\rho} e^{i(\omega t + \phi)} + \cancel{A e^{\alpha_1 t} + B e^{\alpha_2 t}} \to 0$$

When $\gamma > 0$, homogeneous solution decays, so after long time,

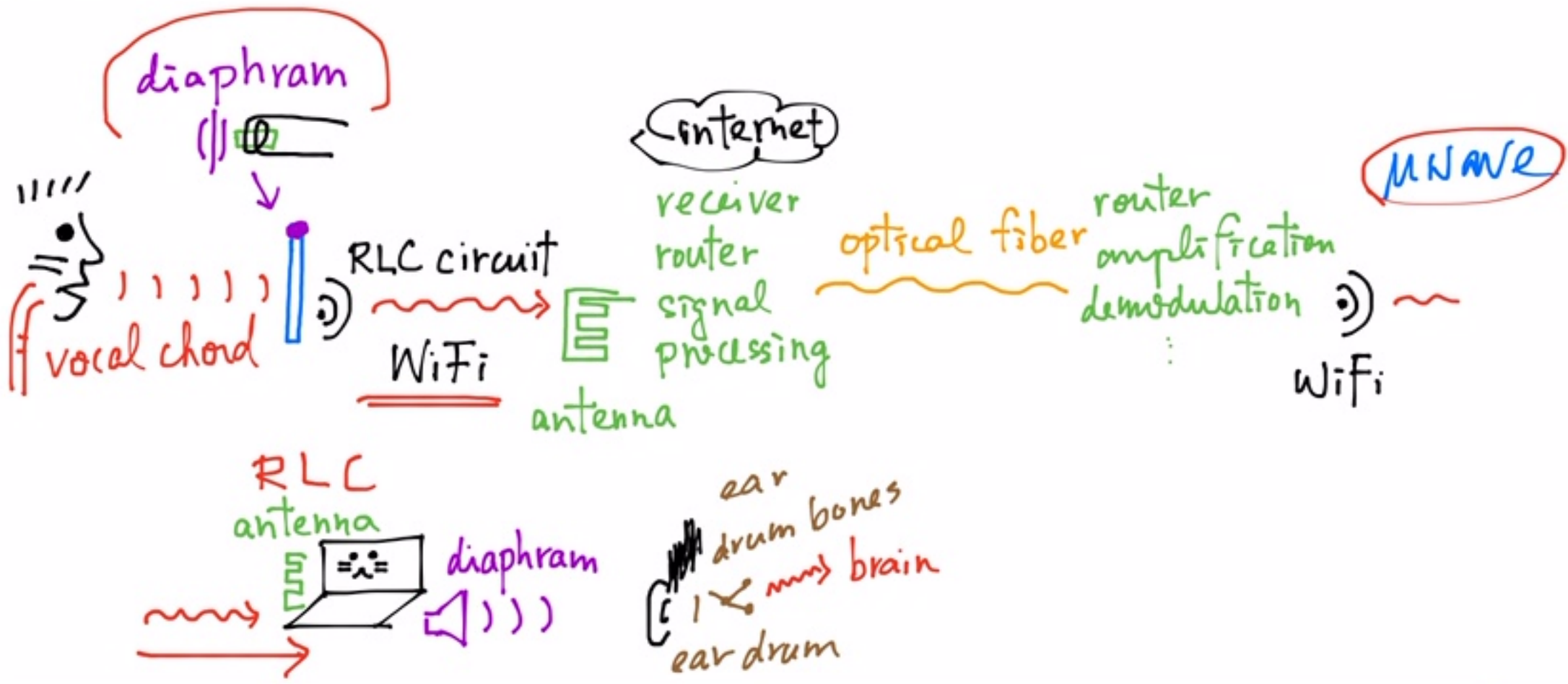
$$x(t) \to x_s(t) = \frac{f}{\rho} \underbrace{e^{i(\omega t + \phi)}}_{\cos(\omega t + \phi)}$$

Lecture 2-2 Driven oscillator (I)

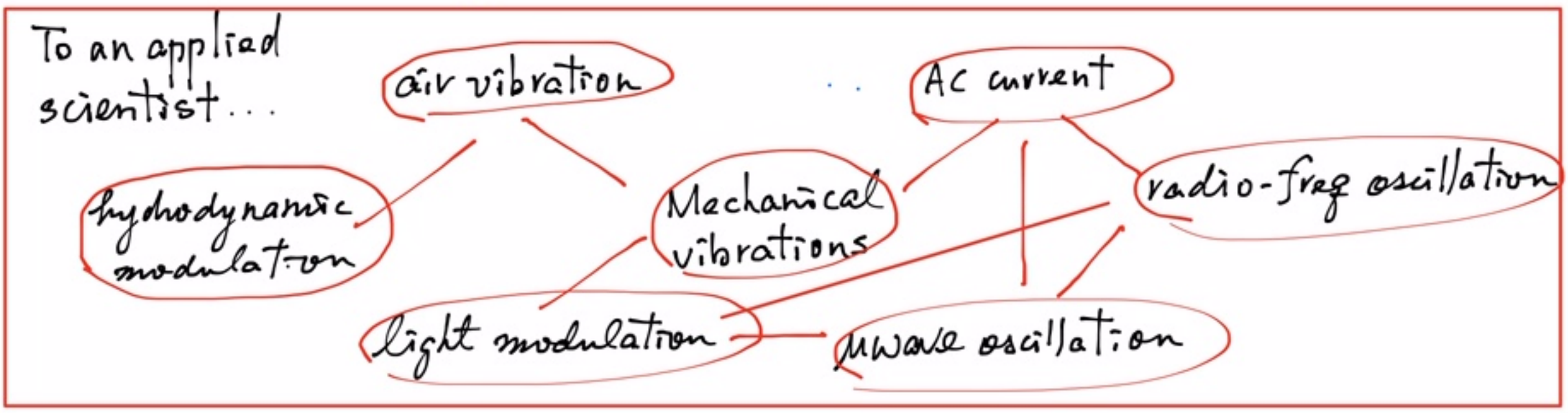
Amp & phase

Demonstration of driven oscillation.

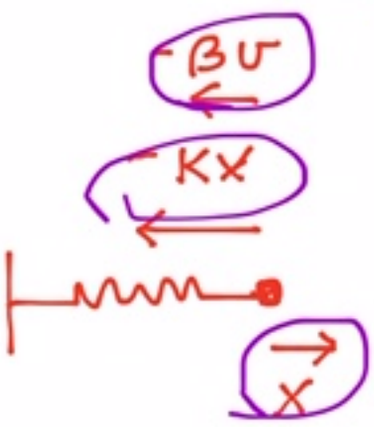
How do you hear me? How does a system respond to external force?



To an applied scientist...



To an (arrogant) physicist, all these are nothing but



$$\text{driving force} = mf\cos(\omega t+\theta) \Rightarrow \boxed{x'' + \gamma x' + \omega_0^2 x = f\cos(\omega t+\theta)}$$

- x: displacement of anything
- γ: friction / damping coefficient
- ω_0: natural freq
- mf: external force
- ω: driving angular freq $= \frac{2\pi}{\text{period}}$

Also $Q = \frac{\omega_0}{\gamma}$ is the quality factor.

First we go to complex notation $\boxed{\tilde{x} = x + iy}$ $x = Re[\tilde{x}]$, x and y

satisfy

$$\boxed{x'' + \gamma x' + \omega_0^2 x = f\cos(\omega t + \theta)} \quad + \quad i \qquad x, y, f \in \text{real}$$

$$\boxed{y'' + \gamma y' + \omega_0^2 y = f\sin(\omega t + \theta)}$$

Combine the 2 eqns, we get $\qquad \tilde{f} \equiv f e^{i\theta}$

$$\tilde{x}'' + \gamma \tilde{x}' + \omega_0^2 \tilde{x} = f e^{i(\omega t + \theta)} \equiv \tilde{f} e^{i\omega t}$$

The physics solution will be $x_{ph} = Re[\tilde{x}]$

$$\tilde{x} = A e^{i\omega t} \Rightarrow A\underbrace{(-\omega^2 + i\gamma\omega + \omega_0^2)}_{\tilde{C}} = \tilde{f}$$

See Lecture 2-1. the general solution is

See Lecture 1

$$\tilde{x} = \tilde{x}_s + \tilde{x}_H \quad \begin{cases} \text{Homogeneous solution } \tilde{x}_H = \tilde{A} e^{\alpha_1 t} + \tilde{B} e^{\alpha_2 t} \\ \text{special solution } \tilde{x}_s = \dfrac{\tilde{f}}{\omega_0^2 - \omega^2 + i\omega\gamma} e^{i\omega t} \end{cases}$$

Real general solution $x = Re[\tilde{x}] = Re[\tilde{x}_s] + Re[\tilde{x}_H]$

$$\equiv x_s + x_H$$

The special solution is $x_s = Re\left[\dfrac{\tilde{f} e^{i\omega t}}{\tilde{C}}\right]$,

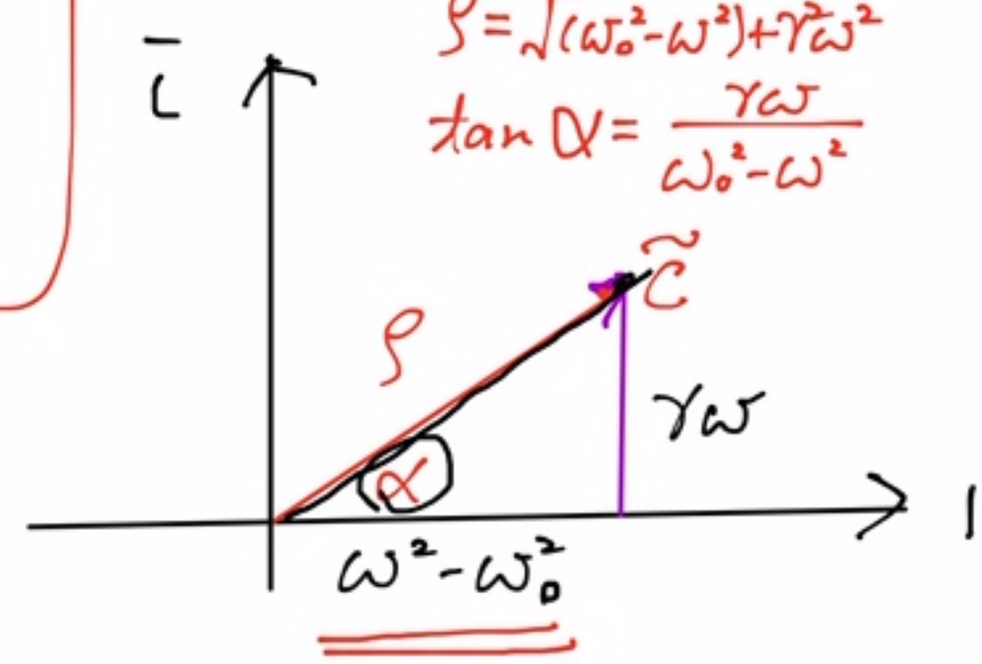
driving freq

$$\tilde{C} \equiv \omega^2 - \omega_0^2 + i\omega\gamma$$

$$\equiv \rho e^{i\alpha}$$

$$\rho = \sqrt{(\omega_0^2 - \omega^2) + \gamma^2\omega^2}$$

$$\tan\alpha = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$



$$\Rightarrow x_s = Re\left[\frac{f e^{i\theta}}{\rho e^{i\alpha}} e^{i\omega t}\right] = \frac{f}{\rho} Re\{e^{i(\omega t + \theta - \alpha)}\}$$

$$\equiv A\cos(\omega t + \theta - \phi) \qquad \phi = -\alpha$$

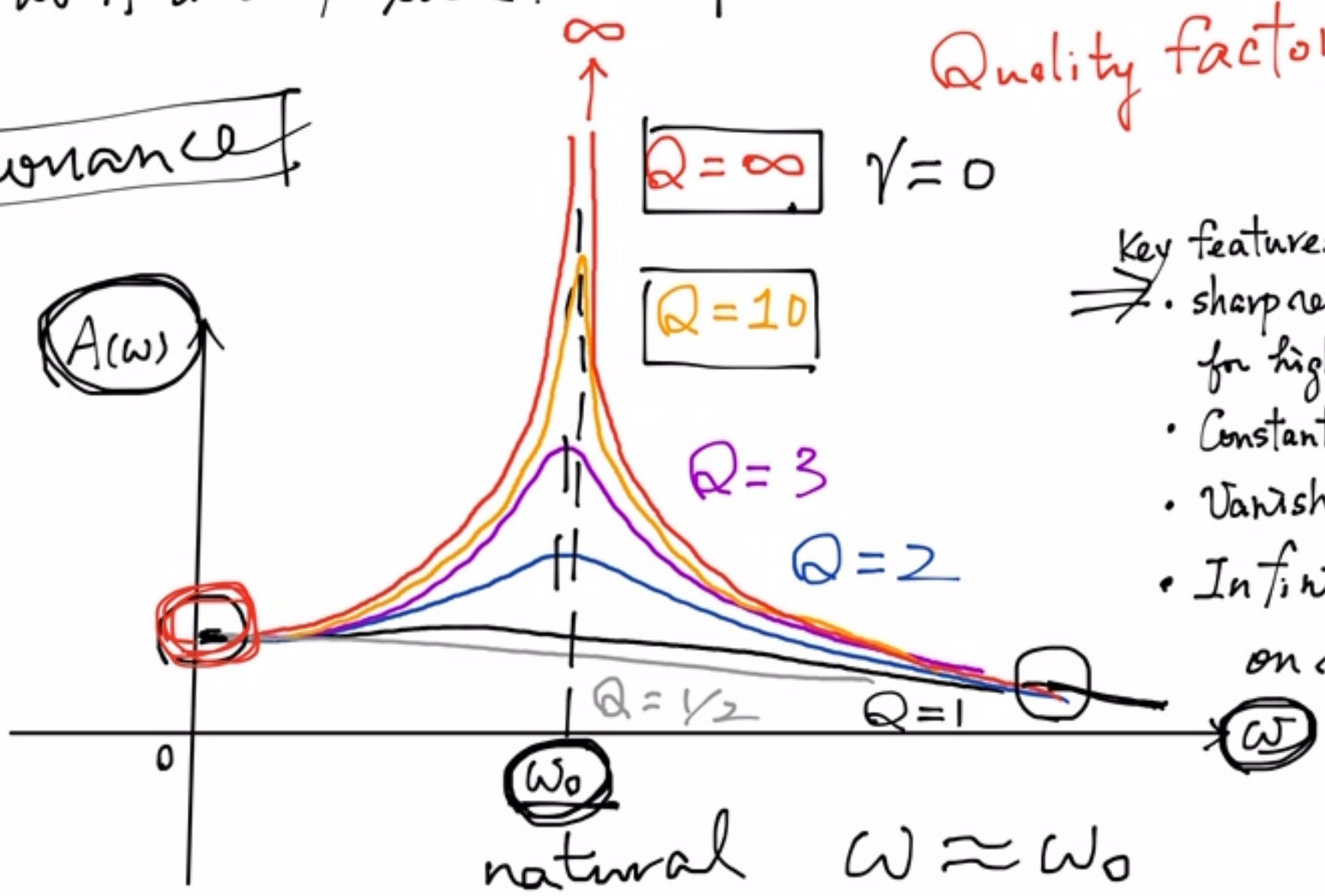
Amplitude $\boxed{A = \dfrac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}}$, phase shift $\phi = -\boxed{\tan^{-1}\dfrac{\gamma\omega}{\omega_0^2 - \omega^2}}$

- θ is set to zero in the textbook, we will follow it now $\theta \equiv 0$
- Both amplitude and phase depend on driving freq ω relative to ω_0.

How do A and ϕ look?

Quality factor $\equiv \boxed{\dfrac{\omega_0}{\gamma}}$

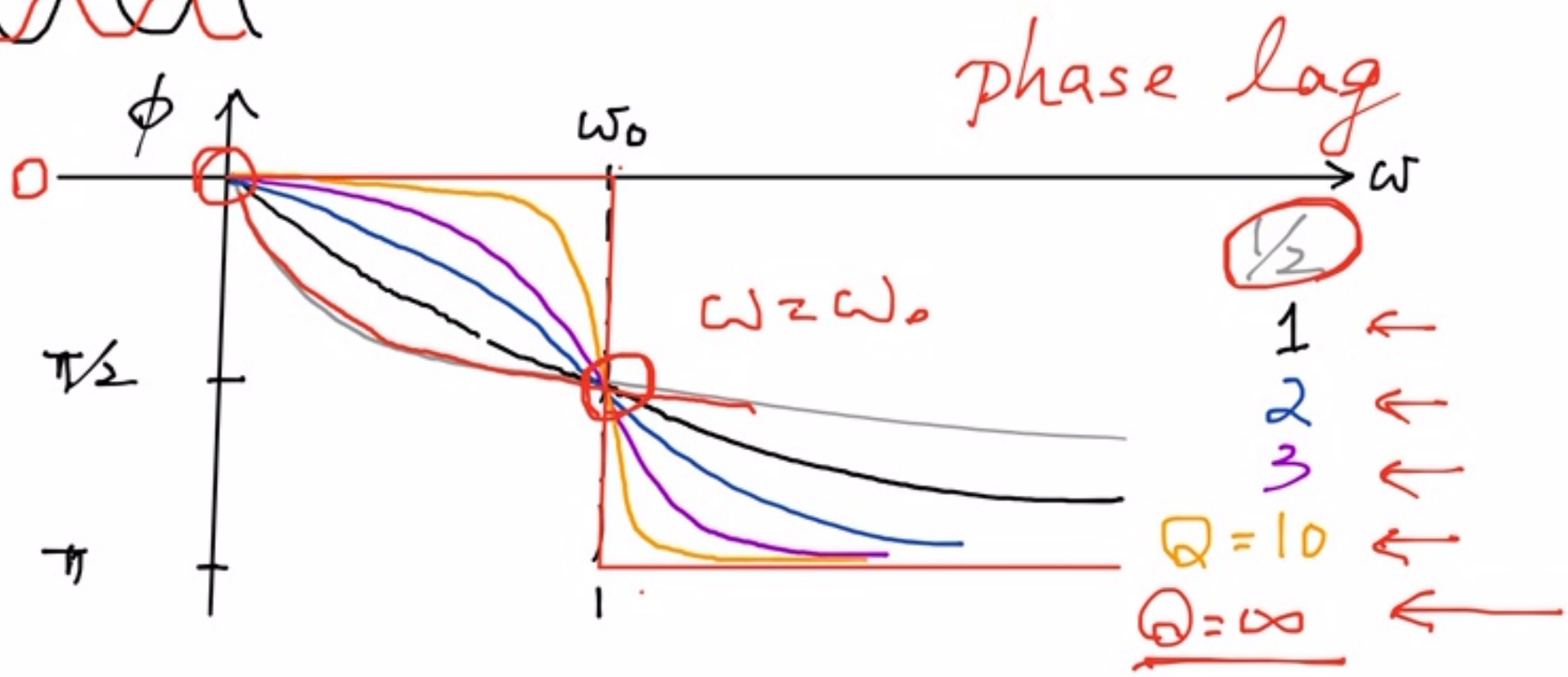
$\boxed{\text{resonance}}$



Key features:

- sharp resonance $\omega \approx \omega_0$ for high Q oscillator
- Constant response at $\omega \to 0$
- Vanishing response $\omega \to \infty$
- Infinite amplitude on resonance when $Q=\infty$ (SHO)

natural $\omega \approx \omega_0$



Key features

- Low driving freq $\omega \to 0$: no phase shift
- Resonance freq $\omega=\omega_0$: $\pi/2$ phase lag $\phi=-\pi/2$
- High freq limit $\omega \gg \omega_0$: π phase lag $\phi=-\pi$
- Higher $Q \Rightarrow$ sharper transition near resonance
- Infinite $Q=\infty$ (SHO) $\Rightarrow$ phase jumps at $\omega=\omega_0$ by $-\pi$.

Let's look at the boring regimes

Case 1: $\omega \ll \omega_0$ low driving freq $\Rightarrow$ $\lim_{\omega\to 0} A = \boxed{\frac{f}{\omega_0^2}}$, $\lim_{\omega\to 0} \phi = 0$

(Hooke's law)

This is static force response.

$F = mf = kx \Rightarrow x = f/\omega_0^2$

Case 2: $\omega \gg \omega_0$ high driving freq $\Rightarrow$ $\lim_{\omega\to\infty} A = \frac{f}{\omega^2} \to 0$, $\lim_{\omega\to\infty} \phi = -\pi$

no response at infinite freq

Very important concept!!

indep of quality of oscillator

$\Rightarrow$ nothing responds to infinite fast modulation!!

Near resonant driving freq

Case 3 near sonance $\omega \simeq \omega_0$.

we define detuning "$\Delta \equiv \omega - \omega_0 \ll \omega_0$"

$\Rightarrow \omega + \omega_0 \approx 2\omega_0$

$\Rightarrow (\omega^2 - \omega_0^2) = 2\omega_0 \Delta$

$\Rightarrow (\omega^2-\omega_0^2)^2 + \gamma^2\omega^2 \simeq \boxed{4\omega_0^2\Delta^2 + \gamma^2\omega_0^2}$

$\approx 4\omega_0^2(\Delta^2 + \gamma^2/4)$

$$\boxed{A = \frac{f}{\sqrt{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}}}$$

$$\approx \frac{f}{2\omega_0}\boxed{\frac{1}{\sqrt{(\omega-\omega_0)^2+\gamma^2/4}}}$$

(You will see this 100x in Physics.)

Peak @ $\omega \approx \omega_0$, $\boxed{A_{peak} = \frac{f}{\omega_0\gamma} = \frac{f}{\omega_0^2}Q}$

Lecture 2-3 Energy transfer in a driven oscillator

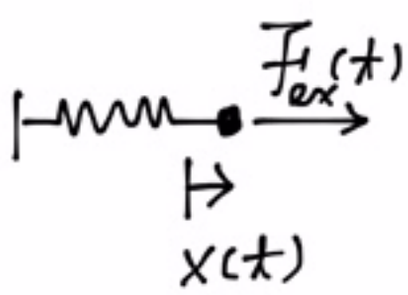
Consider a turkey in a microwave oven, we heat the bird by driving the electrons in the H_2O molecules with microwaves at 2.45 GHz.

1000 W

60% eff

$\omega_0/2\pi = 2.45\,GHz$ is one major sonance in H_2O.

How do we calculate energy transfer?



System gains energy from driving force F_{ex} as

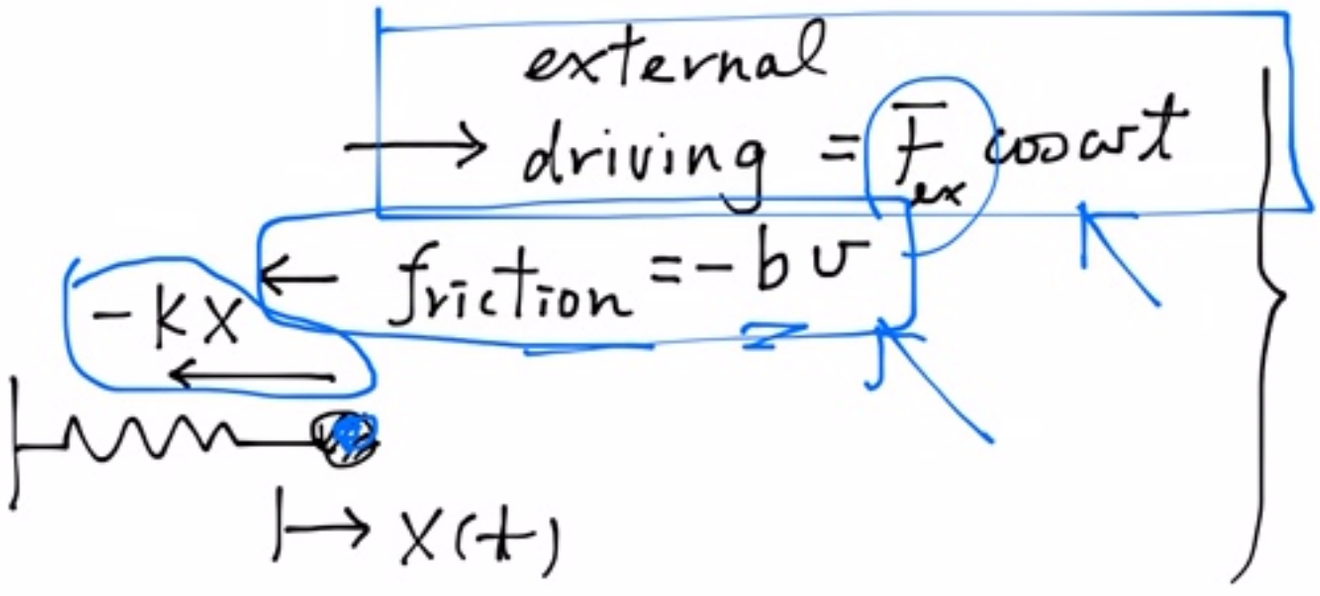
$$\Delta E = \text{Work} = \vec{F}_{ex} \cdot \Delta\vec{x}$$

Power of the driving force

= energy transfer per second

$$P = \frac{\Delta E}{\Delta t} = \vec{F}_{ex} \cdot \frac{\Delta \vec{x}}{\Delta t} = \vec{F}_{ex} \cdot \vec{v}$$

In our driven oscillator, two external forces act on the mass.



Equation of motion

$$x'' + \gamma x' + \omega_0^2 x = f\cos\omega t$$

$\gamma = b/m$. $f = F_{ex}/m$

How do these forces transfer energy to the oscillator?

Energy transfer due to friction:

$$P_{friction} = \boxed{f_{friction}} \cdot v = \boxed{-m\gamma v^2} < 0 \quad \text{dissipation}$$

$\Rightarrow$ friction always takes energy out of system

Given the special solution $\boxed{x(t) = \frac{f}{\rho}\cos(\omega t+\phi)}$ ← Lecture 2-2

$$\boxed{x'(t) = -\frac{f}{\rho}\omega\sin(\omega t+\phi)}$$

$$P_{friction} = -m\gamma\frac{f^2}{\rho^2}\omega^2\boxed{\sin^2(\omega t+\phi)} \le 0 \text{ @ all time}$$

$\sin^2 = 1/2$

Averaging over one cycle: $\boxed{\langle P_{friction}\rangle = -m\gamma\omega^2\frac{f^2}{\rho^2}\boxed{\frac{1}{2}}}$

Not surprisingly, friction dissipates energy, driven or not.

Energy transfer due to external driving.

$$P_{ex} = F_{ex}\cdot v = mf\cos\omega t\left[-\frac{f}{\rho}\omega\sin(\omega t+\phi)\right]$$

$$= -\boxed{\frac{mf^2\omega}{\rho}}\boxed{\cos\omega t\sin(\omega t+\phi)}$$ obviously oscillating!!

$\langle\sin^2\omega t\rangle = 1/2$

$\langle\cos^2\omega t\rangle$

what do we do with this?

math: $\cos\omega t\sin(\omega t+\phi)$

$= \cos\omega t[\sin\omega t\cos\phi + \cos\omega t\sin\phi]$

$= \sin\omega t\cos\omega t\cos\phi + \cos\omega t\cos\omega t\sin\phi$

$= \frac{1}{2}\sin 2\omega t\cos\phi + \cos^2\omega t\sin\phi$

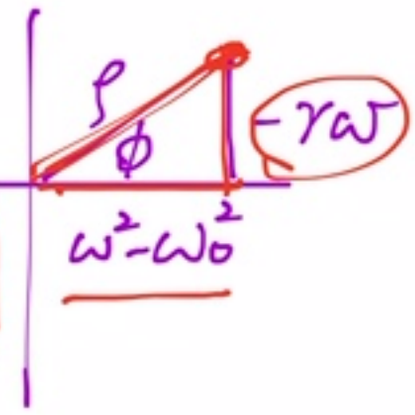
average to zero average to 1/2

$$\langle P_{ex}\rangle = -\frac{mf^2\omega}{\rho}\boxed{\frac{1}{2}}\sin\phi > 0$$

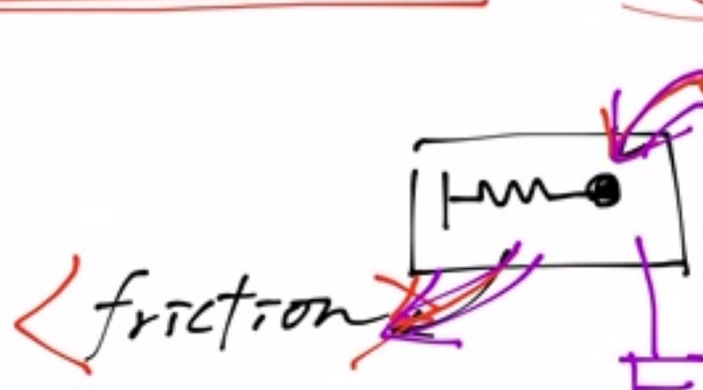
$$= \boxed{\frac{mf^2\gamma\omega^2}{2\rho^2}} = -\langle P_{friction}\rangle !!$$

$$\boxed{\tan\phi = \frac{-\gamma\omega}{\omega^2-\omega_0^2}}$$

$$\Rightarrow \boxed{\sin\phi = \frac{-\gamma\omega}{\rho}}$$



$\langle \text{Total power transfer} \rangle = \langle P_{ex} \rangle + \langle P_{friction} \rangle = 0$



$\boxed{\text{steady state}}$

$E = \text{const.}$

$x = \frac{f}{\rho} \cos(\omega t + \phi)$

Finally total energy in the system E_{tot} = kinetic + potential

$$\boxed{E_{tot} = \frac{1}{2} k x^2 + \frac{1}{2} m v^2 = \frac{m}{2} \omega^2 \frac{f^2}{\rho^2} \boxed{\cos^2(\omega t + \phi)} + \frac{m}{2} \frac{f^2}{\rho^2} \omega^2 \boxed{\sin^2(\omega t + \phi)}}$$

$$= \boxed{m \omega^2 \frac{f^2}{\rho^2}} = \text{const of time}$$

Thus $\langle P_{ex} \rangle = -\langle P_{friction} \rangle = E \boxed{\frac{\gamma}{2}} = \Delta E / \Delta t$

dissipation rate

energy stored in the system

system

E

$E\gamma/2 = \Delta E/\Delta t$ $\boxed{2/\gamma}$ $\gamma/2$ rate

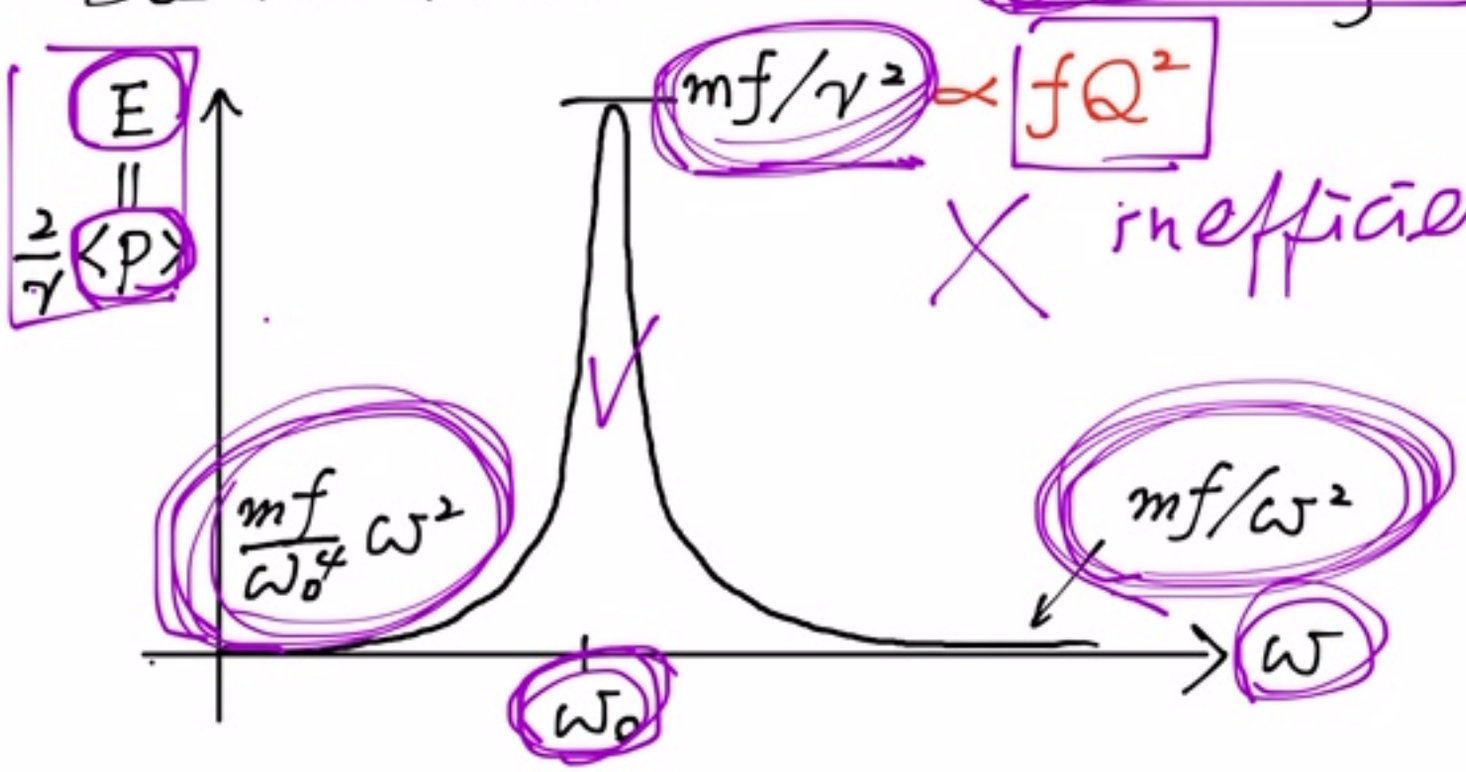
energy input from driving

$-E\gamma/2$

energy dissipated by friction

$\Rightarrow$ the lower γ is the more energy is stored, less energy dissipated and higher quality !! $\boxed{Q = \frac{\omega_0}{\gamma}}$

See how E looks like $E = mf \frac{\omega^2}{\rho^2} = mf \boxed{\frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2}}$



X inefficient

$\omega \ll \omega_0$ $\frac{\omega^2}{\omega_0^4} \propto \omega^2$

$\omega \gg \omega_0$ $\frac{1}{\omega^2}$

$\boxed{\omega \approx \omega_0 \quad \frac{1}{\gamma^2}}$