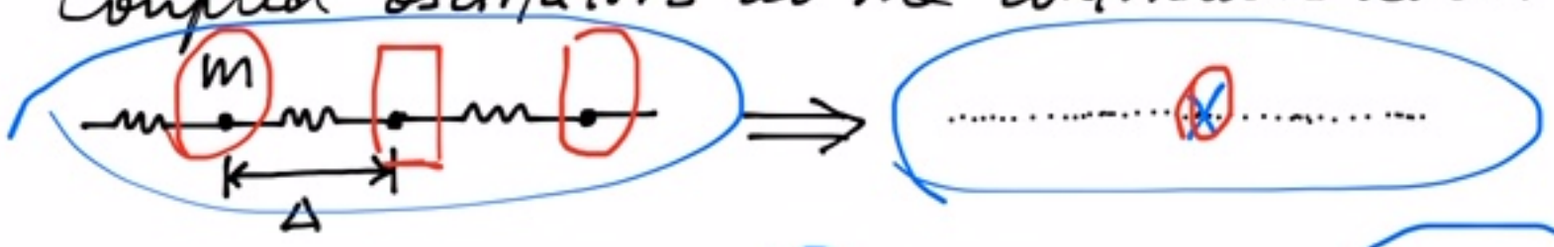
Lecture 5-1 Wave Equation

Cheng Chin

Coupled oscillators in the continuum limit



Δ goes to zero, $m \to \rho\Delta$, $x_n(t) \to \varphi(x,t)$

displacement of the n-th mass

displacement of the particle @ x

$$m x_i'' = k(x_{i-1} - 2x_i + x_{i+1}) \Rightarrow m\partial_t^2\varphi = k[\varphi(x+\Delta) - 2\varphi(x) + \varphi(x-\Delta)]$$

$$\Rightarrow m\partial_t^2\varphi = k\Delta\left[\boxed{\frac{\varphi(x+\Delta)-\varphi(x)}{\Delta}} - \boxed{\frac{\varphi(x)-\varphi(x-\Delta)}{\Delta}}\right]$$

$$\approx k\Delta(\partial_x\varphi(x+\Delta/2) - \partial_x\varphi(x-\Delta/2))$$

$$\approx k\Delta^2\partial_x^2\varphi(x)$$

$f'(t)$

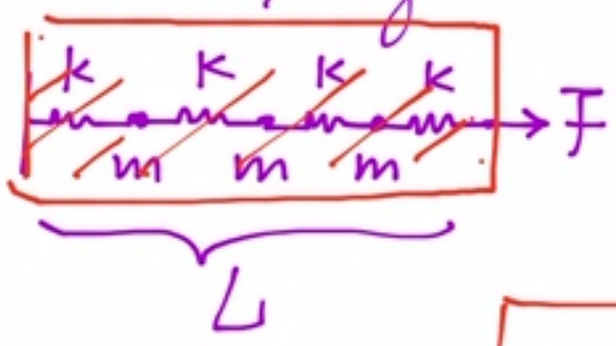
$\frac{\partial}{\partial x}\varphi(x,t)$ $\frac{\partial}{\partial t}\varphi$

$\partial_x\varphi(x,t)$ $\partial_t\varphi$

$\Delta \to 0 \Rightarrow \rho\partial_t^2\varphi = \mathcal{E}\partial_x^2\varphi$. $\rho = m/\Delta$: linear density

$\boxed{\mathcal{E}} = K\Delta$: $\boxed{\text{Young's modulus}}$ (1D)

What's Young's modulus?



$F = kx$

Total length $\boxed{\mathcal{L} = N\Delta}$

Displacement $= \boxed{NF/K} = \delta L$

$\mathcal{L}^{D-1}$

$$\boxed{\mathcal{E} = \frac{\text{stress}}{\text{strain}}} = \frac{\text{Pressure}}{\text{fractional deformation}} = \frac{F/A}{\delta L/L} = K\Delta \ (1D)$$

Example: Rubber $\mathcal{E} = 0.01$ $\boxed{\text{GPa}}$ $(1\,\text{Pa} = \text{N}/\text{m}^2)$

Wood $= 10 \times 10^9$ Pa

Cu $= 100$

Diamond $= 1000$ $\Rightarrow$ $\boxed{\text{Stiffness } \mathcal{E} = \text{tensile strength}}$.

Wave equation $\boxed{\rho\partial_t^2\varphi = E\partial_x^2\varphi}$ or $\partial_t^2\varphi = v^2\partial_x^2\varphi$ $\boxed{v \equiv \sqrt{E/\rho}}$ $\Big\}\ \omega = \sqrt{K/m}$

(by d'Alembert & Euler, 1746)

How do we solve it? $X = Ae^{i\omega t}$

Old method: $\varphi(x,t) = \boxed{A(x)}\,\boxed{e^{i\omega t}}$

Just like we did $x = Ae^{\alpha t}$ or $\vec{x} = \vec{A}e^{\alpha t}$

$\Rightarrow \partial_t \varphi = i\omega\varphi,\ \partial_t^2 \varphi = -\omega^2\varphi,\ \partial_x \varphi = A' e^{i\omega t},\ \partial_x^2 \varphi = A'' e^{i\omega t}$

$\Rightarrow -\omega^2 A e^{i\omega t} = v^2 A'' e^{i\omega t} \Rightarrow \boxed{A'' = -\frac{\omega^2}{v^2} A}$

$e^{i\omega t}$ constant

Next we assume $A(x) = e^{ikx} \Rightarrow \boxed{-k^2 = -\omega^2/v^2} \Rightarrow \boxed{\omega = kv}$

$\Rightarrow$ Solution is $\varphi(x,t) = (Ae^{ikx} + Be^{-ikx})(Ce^{i\omega t} + De^{-i\omega t})$

or any equivalent forms

k for ω

k: wave #

$\omega = kv$

$\Rightarrow$ general solution includes all superpositions with $\boxed{\omega_j = k_j v}$

$$\varphi = \sum_j A_j e^{ik_j(x-vt)} + B_j e^{-ik_j(x-vt)} + C_j e^{ik_j(x+vt)} + D_j e^{-ik_j(x+vt)}$$

$$= \sum_j A_j \cos k_j(x-vt) + B_j \sin k_j(x-vt) + C_j \cos k_j(x+vt) + D_j \sin k_j(x+vt)$$

Finally, we need the B.C.s to determine the coefficients.

Say, the initial position and velocity of all oscillators $\varphi(x, t=0)$ & $\partial_t \varphi(x, t=0)$

Next lecture on Fourier transform we will show all functions can be written in the form of $f(x) = \int e^{-ikx} f(k)$

Method 2: General solution of the form $\varphi(x,t) = f(x-vt)$ $\quad f(x+vt)$

$f(u)$

Back to $\boxed{\partial_t^2 \varphi = v^2 \partial_x^2 \varphi}$, we can introduce $\boxed{u = vt}$

$\boxed{\partial_u^2 \varphi = \partial_x^2 \varphi}$ so as long as the function looks similar when we exchange x and u would work

How about

$\varphi = x+u,\ x-u,\ f(x\pm u)$, $\cancel{f(xu)}$, $\cancel{f(x^2+u^2)}$? $\quad e^{x+u}$

$f(x)$ $\quad x-vt$

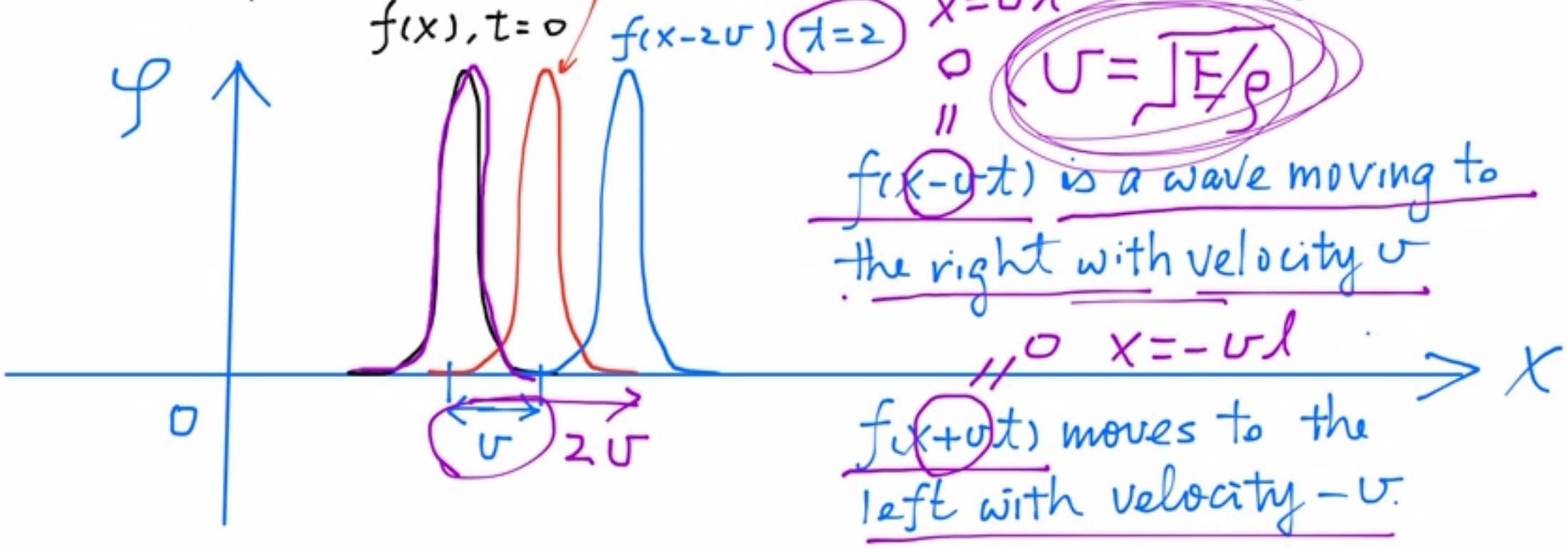
Let's check $\varphi = f(x \pm vt) \Rightarrow \partial_x^2 f(x\pm vt) = f''(x\pm vt)$

$\partial_t^2 f(x\pm vt) = v^2 f''(x\pm vt)$

$\boxed{\text{Any function } f(x\pm vt) \text{ works!!}}$

How about $\cancel{f(xt)}$? $\partial_x^2 f = t^2 f''(xt)$, $\partial_t^2 f = x^2 f''(xt)$ $\quad \cancel{f(x^2+u^2)}$

Let's check out $\varphi = f(x-vt)$ for some general $f(x)$



$x = vt$

$\parallel$

$f(x-vt)$ is a wave moving to the right with velocity v

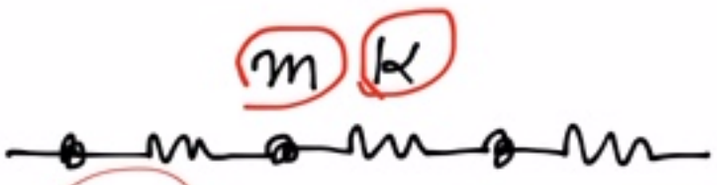
$v = \sqrt{E/\rho}$

$\parallel$ $x = -vt$

$f(x+vt)$ moves to the left with velocity $-v$.

The wave equation supports all functions that propagates in any direction, with any form at velocity $v = \sqrt{E/\rho}$.

5-2 Summary of discrete and continuous waves.



$$X_n'' = \omega^2 (X_{n+1} - 2X_n + X_{n-1})$$

$$X_n = \sum_j A_j e^{i(n\epsilon_j + \omega_j t)} + B_j e^{i(n\epsilon_j - \omega_j t)} + C_j e^{i(-n\epsilon_j + \omega_j t)} + D_j e^{i(-n\epsilon_j - \omega_j t)}$$

$$\boxed{\omega_j = 2\omega_0 \sin\tfrac{1}{2}\epsilon_j}$$

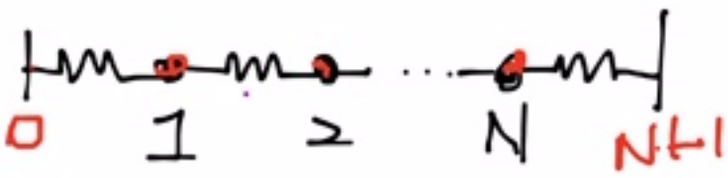
$$\boxed{\omega_0 = \sqrt{k/m}}$$

$$\rho \partial_t^2 \varphi = \mathcal{E} \partial_x^2 \varphi \quad \text{W eqn}$$

$$\varphi(x,t) = \sum_j A_j e^{i(k_j x + \omega_j t)} + B_j e^{i(k_j x - \omega_j t)} + C_j e^{i(-k_j x + \omega_j t)} + D_j e^{i(-k_j x - \omega_j t)}$$

$$\boxed{\omega_j = v k_j} \quad \boxed{v = \sqrt{\mathcal{E}/\rho}}$$

For a string of m oscillators



Solutions are

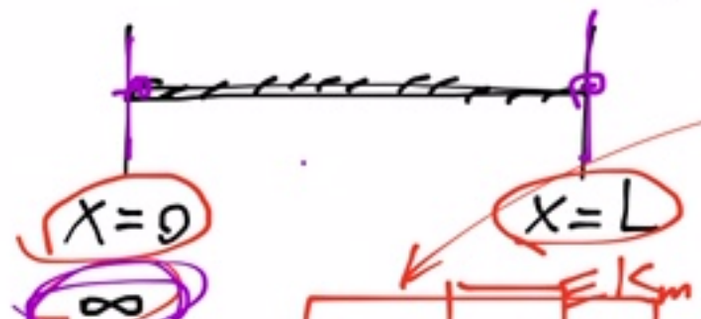
$$X_n = \sum_{m=1}^{N} A_m \sin\frac{nm\pi}{N+1} \cos(\omega_m t + \phi_m)$$

$$\Rightarrow X_0 = X_{N+1} = 0, \quad \boxed{\omega_m = 2\omega_0 \sin\frac{m\pi}{2(N+1)}}$$

Boundary conditions

$X_n(0), X_n'(0)$

a continuous string



cos does not work!

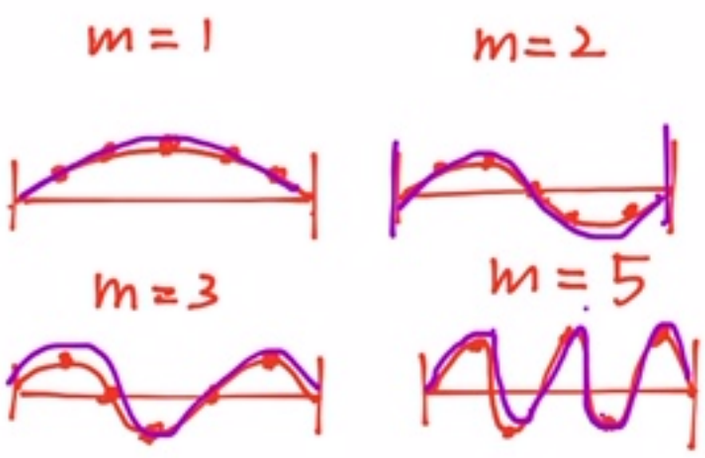
$$\varphi(x) = \sum_{m=1}^{\infty} A_m \sin\frac{m\pi x}{L} \cos(\omega_m t + \phi_m)$$

$$\varphi(x=0) = \varphi(x=L) = 0$$

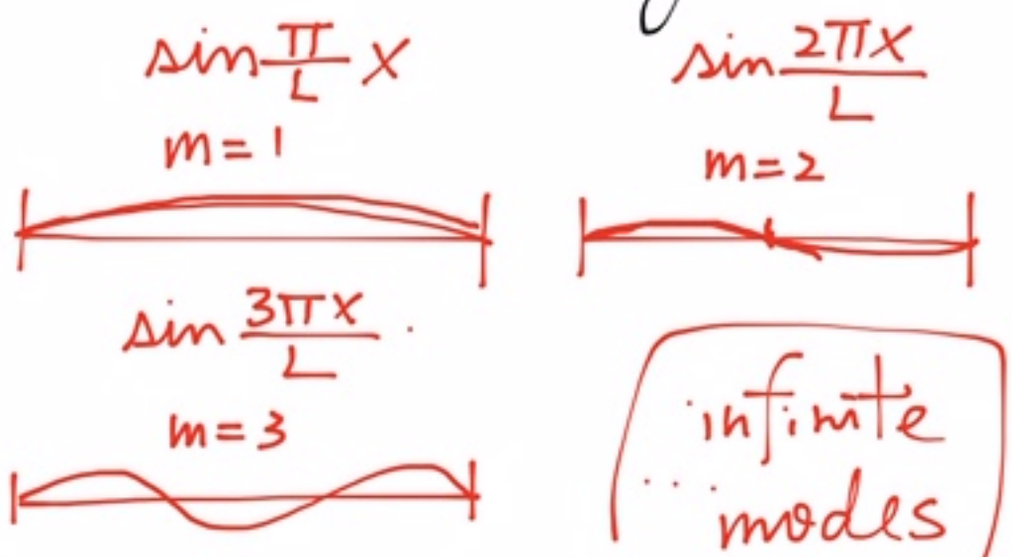
$$k_m = \frac{m\pi}{L}, \quad \omega_m = k_m v$$

$\boxed{\varphi(x, t=0)}$ & $\boxed{\partial_t \varphi(x, t=0)}$

position velocity.

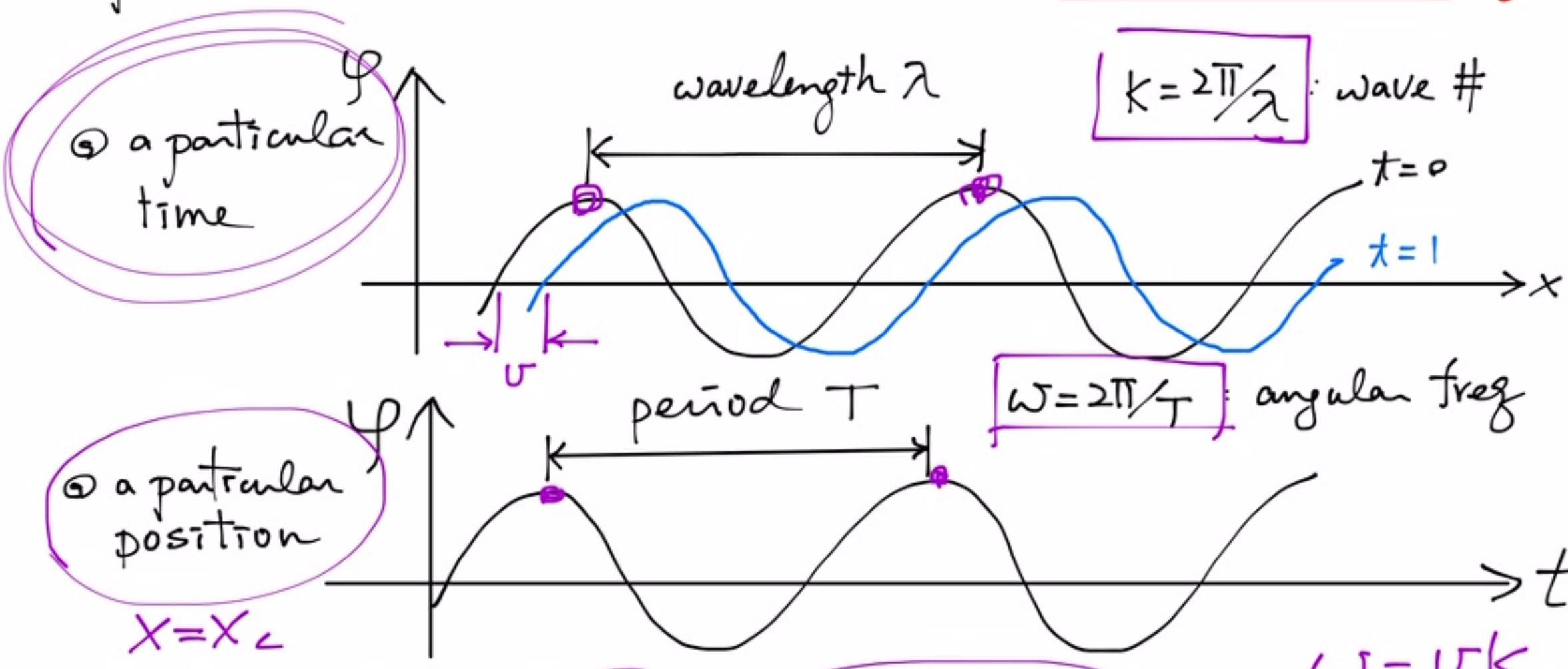


5 particles
⇓
5 modes



infinite modes

In particular, for sinusoidal functions $\varphi = \cos k(x - vt) = f(x - vt)$

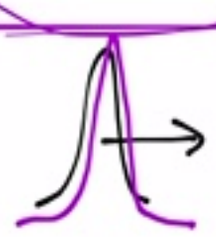


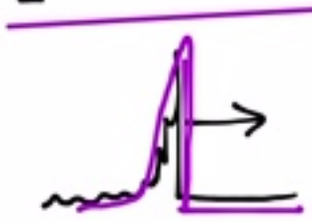
$\omega = vk$

$$\boxed{\text{velocity} = \omega/k = \text{wavelength}/\text{period}}$$

All kinds of waves: $\cos(kx - \omega t)$ or $\sin(kx - \omega t)$ called sinusoidal travelling waves

$\boxed{\cos kx \cos \omega t} = \frac{1}{2}\cos(kx - \omega t) + \frac{1}{2}\cos(kx + \omega t)$: standing waves


solitary waves


shock waves

Wave eqn $\partial_t^2 \varphi = v^2 \partial_x^2 \varphi$ describes those that remain the wave form and travels at a particular speed $\boxed{v = \sqrt{E/\rho}}$

v is higher for higher modulus (stiffness) & lower mass density

$$\boxed{\rho \quad E}$$

air $\rightarrow$ ρ small, $\frac{300\,m}{s}$

steel $\rightarrow$ E large $\frac{1\,km}{s}$

Consider we have $\boxed{\varphi(x, t=0)} = \boxed{\phi(x)}$, $\partial_t \varphi(x, t=0) = 0$. $\phi(0) = \phi(L) = 0$

$$\varphi(x,t) \equiv \sum_{m=1} \sin k_m x \,(A_m \cos \omega_m t + B_m \sin \omega_m t)$$

$$\partial_t \varphi(x,t) = \sum_{m=1} \sin k_m x \,(-A_m \omega_m \sin \omega_m t + B_m \omega_m \cos \omega_m t)$$

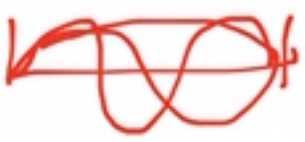
$$\Rightarrow \phi(x) = \sum_{m=1} A_m \sin k_m x$$

$$0 = \sum_{m=1} B_m \omega_m \sin \omega_m x$$

$\Rightarrow$ How do we determine A_m & B_m?

(Fourier Series)

Fourier realized that $\sin k_n x$ and $\sin k_m x$ are orthorgonal.



$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = \frac{L}{4} \boxed{\delta_{nm}}$$

$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx = \frac{L}{4} \delta_{nm}$$

$$\int_0^L \cos \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = 0$$

$\delta_{nm} = 1$ when $n = m$
$\quad\;\; = 0$ when $n \neq m$

Thus we can multiply $\boxed{\sin \frac{n\pi x}{L}}$ on both sides of the eqn, integrate

$$\Rightarrow \int_0^L \phi(x) \sin \frac{n\pi x}{L}\, dx = \sum_{m=1} A_m \boxed{\int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx} = \sum_{m=1} A_m \boxed{\delta_{mn} \frac{L}{4}}$$

$$= \boxed{\frac{L}{4} A_n} \Rightarrow A_n = \frac{4}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L}\, dx$$

$m = n$

$\forall n$

$$\Rightarrow \int_0^L 0 \sin \frac{n\pi x}{L}\, dx = \sum_{m=1} B_m \delta_{mn} \frac{L}{4} = \boxed{\frac{L}{4} B_n} \Rightarrow B_n = 0 \quad \forall n$$

So we solved the problem

$$\varphi(x,t) = \sum_{m=1} A_m \sin \frac{m\pi x}{L} \cos \frac{m\pi}{L} v t \,. \quad (\omega_m = k_m v, \; v = \sqrt{E/\rho})$$

$$A_n = \frac{4}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L}\, dx$$