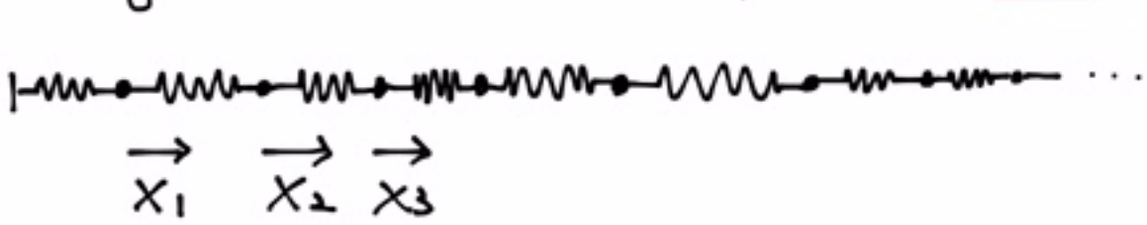
1D Longitudinal waves: particles move in the same direction as the wave form.



Examples: Sound wave, pressure waves, gravitational waves

Transverse waves: particles moves in the perpendicular direction(s)

Examples: strings. ropes, electromagnetic waves,

Mixed waves: water waves. human waves. seismic waves.

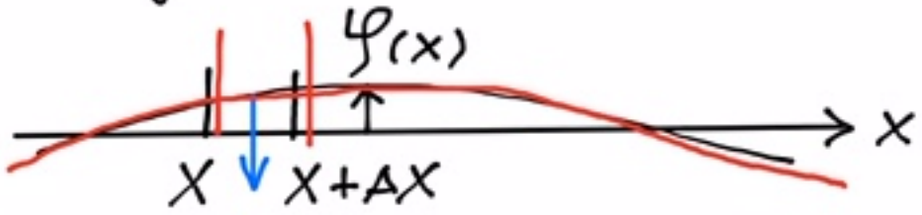
brain waves(??) - hand waves(??)

string waves & ZSM waves are very different!

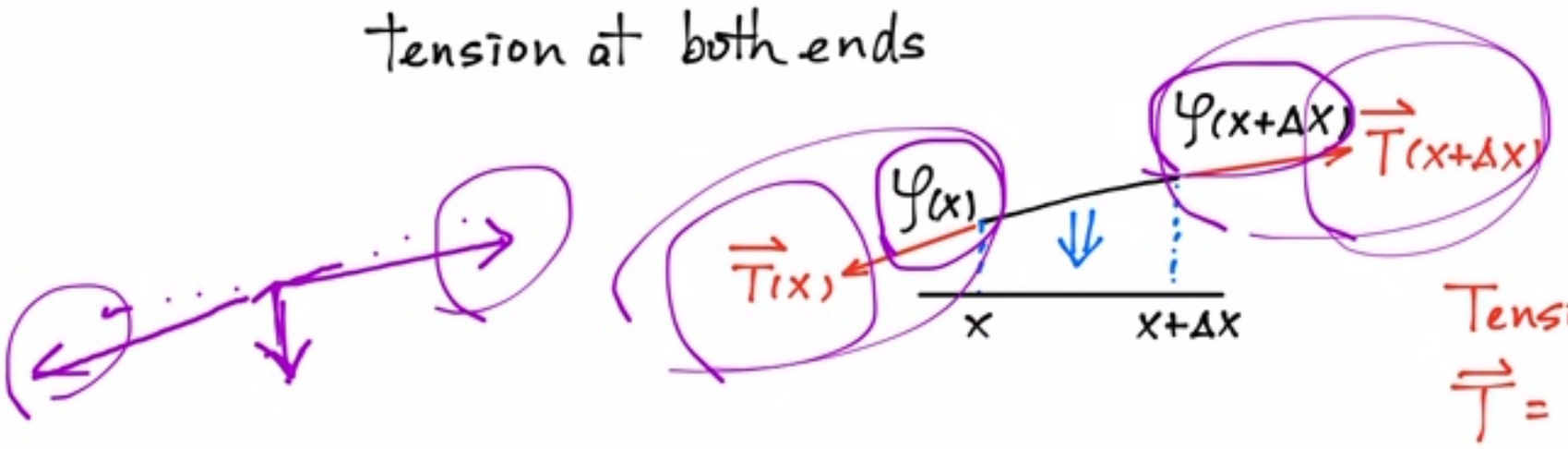
String Waves

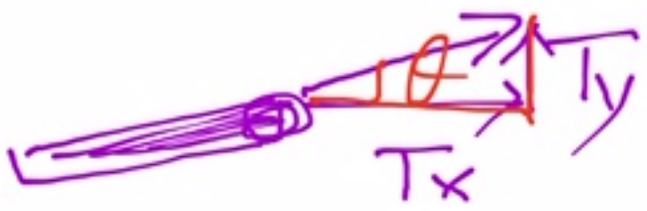
Think about any string instrument.

Why does it oscillate? Does it follow wave eqn?



Consider a small section Δx, the restoring force comes from the tension at both ends





Tension is along the string

$\vec{T} = (T_x, T_y) \Rightarrow \boxed{\varphi'(x) = T_y / T_x}$

Restoring force is in the y-direction:

$F_x = T_x(x+\Delta x) - T_x(x) = 0$

$F_y = T_y(x+\Delta x) - T_y(x) = ?$

Combining all eqns: $T_y = T_x \varphi'(x)$

$T_y(x+\Delta x) - T_y(x) = \frac{d}{dx} T_y(x) \Delta x = \frac{d}{dx} T_x \varphi'(x) \Delta x = T_x \varphi''(x) \Delta x$

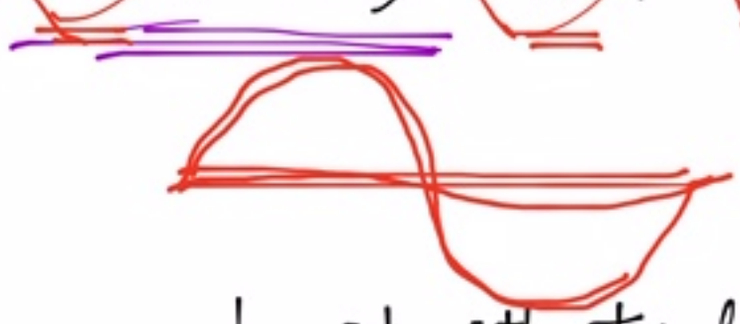
$F = ma \Rightarrow T_x \partial_x^2 \varphi(x,t) \Delta x = \rho \Delta x \partial_t^2 \varphi(x,t)$

ρ: linear density

Viola! $\rho \partial_t^2 \varphi(x,t) = T \partial_x^2 \varphi(x,t)$ is the wave eqn.

I assume $T_x \approx T$, true when $\partial_x \varphi(x,t) = T_y / T_x$ is small. (small amp.)

If not, $T = \sqrt{T_x^2 + T_y^2} = T_x \sqrt{1 + (\partial_x \varphi)^2}$ ⇒ becomes a nonlinear PDE, which is too cool to cover in P143b. ∴ But maybe I'll.



So again we can solve it with trial function like

$(A \cos kx + B \sin kx)(C \cos \omega t + D \sin \omega t)$ or $e^{\pm ikx} e^{\pm i\omega t}$

$A f(x - vt) + B g(x + vt)$

What new thing can we do?

Fourier transform $\varphi(x,t) = \int \varphi(k,t) e^{ikx} dk$ (it's hot.)

Do it again $= \int \left[\int \varphi(k,\omega) e^{i\omega t} d\omega \right] e^{ikx} dk$

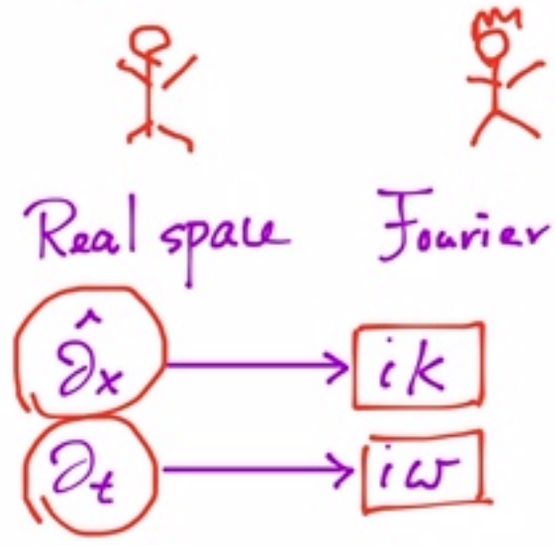
$= \iint \varphi(k,\omega) e^{i(kx+\omega t)} dk\, d\omega$ 2D FT transf.

Can we Fourier transform the whole PDE?

if $\varphi(x) = \int \varphi(k) e^{ikx} dk$

$\varphi'(x) = \int ik \varphi(k) e^{ikx} dk$

$\varphi'(t) = \int i\omega \varphi(\omega) e^{i\omega t} dt$



Differentiation in x/t
=
Multiplication in k/ω

$\Rightarrow \partial_t^2 - v^2 \partial_x^2 \varphi(x,t) \Rightarrow -\omega^2 + v^2 k^2 \varphi(k,\omega) = 0$

$-\omega^2$ $-k^2$ $\varphi(k,\omega)$

Fourier transform reduces diff. Eqn. into algebraic Eqn.

For non-zero $\varphi(k,\omega) \neq 0$ $\omega = vk$, this is exactly our conclusion last week.

So we can always work in the Fourier space, do algebra, then go

back to real space by inverse Fourier transform

$$\psi(x,t) = \iint \varphi(k,\omega)\, e^{i(kx+\omega t)}\, dk\, d\omega$$

An example: $\partial_t^2 \varphi(x,t) + \beta \partial_t \varphi(x,t) = v^2 \partial_x^2 \varphi + f(x,t)$

Damped, driven wave eqn.

My eqn is $(\omega^2 - v^2k^2)\varphi = 0$

Enter Fourier space

$\varphi(x,t)$	$\to$	$\varphi(k,\omega)$
∂_x	$\to$	ik
∂_t	$\to$	$i\omega$

$$\Rightarrow (-\omega^2 + i\beta\omega + v^2k^2)\,\varphi(k,\omega) = f(k,\omega)$$

$$\Rightarrow \varphi(k,\omega) = \frac{f(k,\omega)}{v^2k^2 - \omega^2 + i\beta\omega}$$

$$\Rightarrow \varphi(x,t) = \int \frac{f(k,\omega)}{v^2k^2 - \omega^2 + i\beta\omega}\, e^{i(kx+\omega t)}\, d\omega\, dk$$ This is it.

In general, if you have ANY diff. eqn $D(\partial_x, \partial_t \cdots)\varphi = 0$, where D is some function of all operators $\overset{\hat{\mathcal{F}}}{\Longrightarrow} D(ikx, i\omega t \cdots)\tilde{\varphi} = 0$. ← Fourier transform of φ

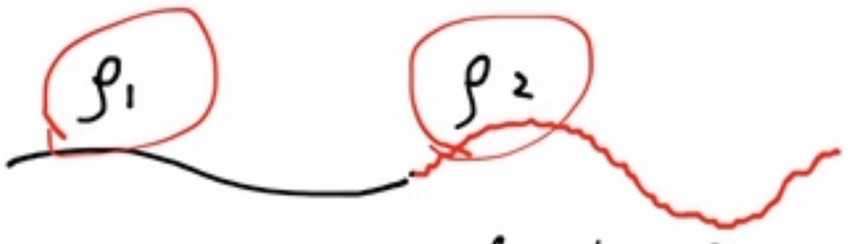
$$\partial_x^2 \varphi_1 + \partial_t \varphi_2 = f_1 \qquad \Rightarrow \quad -k^2\tilde{\varphi}_1 + i\omega\tilde{\varphi}_2 = \tilde{f}_1$$

$$\partial_x^3 \varphi_1 + \partial_t^2 \varphi_2 = f_2 \qquad -ik^3\tilde{\varphi}_1 - \omega^2\tilde{\varphi}_2 = \tilde{f}_2$$

$$\Downarrow$$

$$\varphi_1 = \int \tilde{\varphi}_1\, e^{i(kx+\omega t)}\, d\omega\, dt \quad <= \quad \tilde{\varphi}_1, \tilde{\varphi}_2$$

Example



Connecting two kinds of strings. (Why is this important?)

same for sound waves

)))))))) | ,,,,,,

It is when we talk about EM waves or matter waves.

Consider a wave in medium 1

is propagating to the right $e^{ik_1(x-v_1t)}$ $k_1>0$ $v_1>0$.

$\omega = k_1 v_1$, what will happen in medium 2?

Physicists: Consider special cases: if $\rho_2 \gg \rho_1$, you hit a wall, wave reflected.

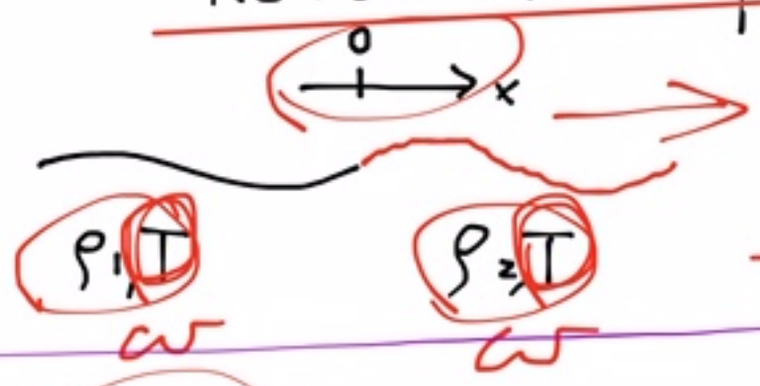
if $\rho_2 = \rho_1$, nothing happens at interface.

$\rho_2 \ll \rho_1$?? wave just goes through

Conclusion: some reflected, some transmitted. How much

Ref and Ted depend on the materials

Math:



Left string is driving the right string

First, $T_1 = T_2 \equiv T$ (near equilibrium) $\Rightarrow$ Left $\varphi(x<0) \sim e^{i(kx-\omega t)} + e^{i(-kx-\omega t)} + \cdots$

Right $\varphi(x>0) \sim e^{i(k_2 x-\omega t)} + \cdots$

They share the same freq $\omega = v_1 k_1 = v_2 k_2 \Rightarrow k_1\sqrt{T/\rho_1} = k_2\sqrt{T/\rho_2}$

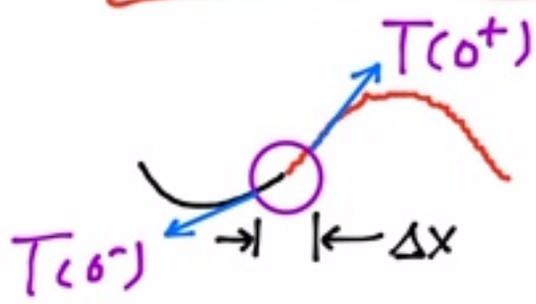
$\Rightarrow k_2 = k_1\sqrt{\rho_2/\rho_1}$

$\frac{k_2}{k_1} = \sqrt{\frac{\rho_2}{\rho_1}}$

2nd, strings are connected $\varphi_1(0^-) = \varphi_2(0^+)$.

But this is still not enough to fully determine the right string.

3rd, $\varphi_1'(0^-) = \varphi_2'(0^+)$ continuity of the slope



The two slopes must approach each when $\Delta x \to 0$ since the mass in the circle $\to 0$ Otherwise $a = F/m \to \infty$.

$\vec{T} \propto (1, d\varphi/dx)$ $\varphi'(0^-) = \varphi'(0)$

Lecture 7-2 propagation of waves — Cheng Chin

Physical meaning of wave # $k = 2\pi/\lambda$ — and propagation direction!

$\varphi = e^{ikx}e^{i\omega t}, \; e^{-ikx}e^{i\omega t}, \; e^{ikx-i\omega t}, \; e^{-ikx-i\omega t}$

$e^{ikx}e^{i\omega t}$: $k(x+vt)$ ←

$e^{-ikx}e^{i\omega t}$: $-k(x-vt)$ →

$e^{ikx-i\omega t}$: $k(x-vt)$ →

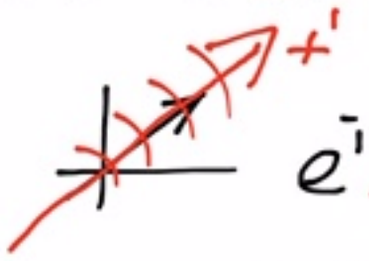
$e^{-ikx-i\omega t}$: $-(k+vt)$ ←

Two go to the left, two to right, just 2 kinds.?

two give $\cos k(x-vt)$ & $\sin k(x-vt)$ ⟵ left going

two give $\cos k(x+vt)$ & $\sin k(x+vt)$ ⟶ right going

If the wave propagates in 2D ⇒



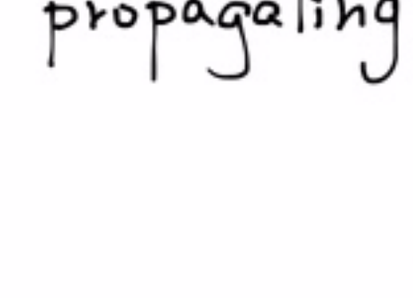
$e^{i(k_x x + k_y y)}e^{-i\omega t}$

$\Rightarrow e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$\Rightarrow \vec{K} = (k_x, k_y, k_z)$ also defines the propagating direction ⇒ convention: $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ is the traveling wave going in the direction of $\vec{K}$ with wave # $k \equiv |\vec{K}| = 2\pi/\lambda = \omega/v$.

So we use $e^{i(\pm kx-\omega t)}$ freely for trial func. $e^{i(\mp kx+\omega t)}$ have the same direction (same real part, btw)

Prove $\cos k(x-vt)$ carries energy to the right



$T \propto \frac{dy}{dx}$

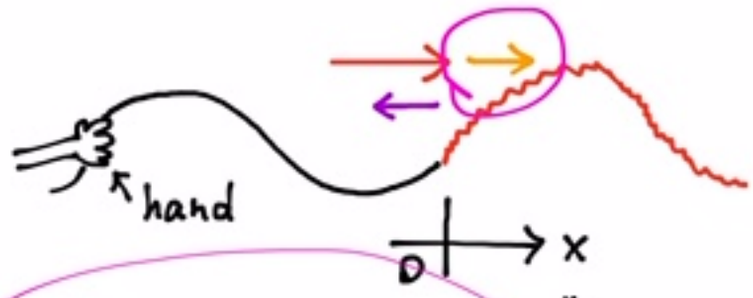
Consider the wire to the left of x exerting a force $-T(1, \partial_x\varphi)$. The work it does to the wire on the right per time is power $P = F\cdot v = F_y v_y = -T\partial_x\varphi\,\partial_t\varphi$

$= -T(-k)(kv)\sin^2 k(x-vt) = Tk^2v\sin^2 > 0 \Rightarrow \longrightarrow \mathcal{E}$

If you assume $\varphi = e^{i(kx-\omega t)}$, be very careful how to do the calculation:

$\Rightarrow P = -T\partial_x\varphi^*\,\partial_t\varphi = -T(-ik)(-i\omega)|\varphi|^2 > 0$ $\quad \int f^* f\,dx$

If we drive the string from the left



coordinate.

We expect $\varphi(x<0) = \underbrace{A e^{i(k_1 x-\omega t)}}_{\text{incident}} + \underbrace{B e^{i(-k_1 x-\omega t)}}_{\text{reflected}}$

$\varphi(x>0) = \underbrace{C e^{i(k_2 x-\omega t)}}_{\text{transmitted}} + \underbrace{0\, e^{i(k_2 x-\omega t)}}_{\text{no reflection for infinite wire.}}$

Boundary condition @ $x=0,\ t=0$. $\varphi(0^-)=\varphi(0^+) \Rightarrow A+B=C$

$\partial_x\varphi(0^-)=\varphi(0^+) \Rightarrow Ak_1 - Bk_1 = Ck_2$

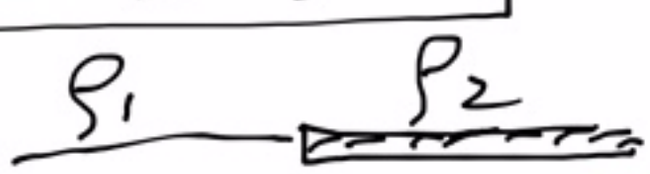
$$\Rightarrow \begin{cases} A+B=C \\ A-B=C(k_2/k_1) \end{cases}$$

$$\Rightarrow C=\frac{2k_1}{k_1+k_2}A,\quad B=\frac{k_1-k_2}{k_1+k_2}A$$

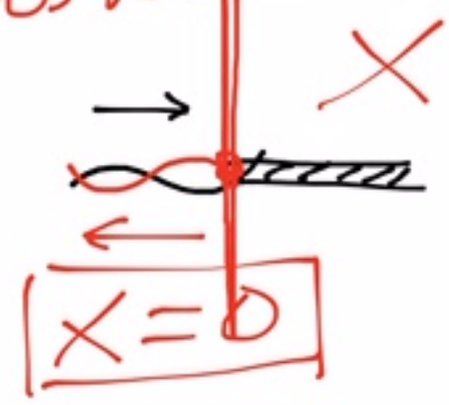
$\omega/k_1 = \sqrt{T/\rho_1}$ $\omega/k_2 = \sqrt{T/\rho_2}$

$\Rightarrow k_1 : k_2 = \rho_1^{1/2} : \rho_2^{1/2}$

$$\Rightarrow C=\frac{2\sqrt{\rho_1}}{\sqrt{\rho_1}+\sqrt{\rho_2}}A,\quad B=\frac{\sqrt{\rho_1}-\sqrt{\rho_2}}{\sqrt{\rho_1}+\sqrt{\rho_2}}A$$



$\cos k(x-vt)$
$+\cos k(x+vt)$



Double check: When $\rho_2 \gg \rho_1 \Rightarrow B=-A,\ C=0$

$R=|B/A|^2 = 100\%$

When $\rho_1=\rho_2 \Rightarrow B=0$. Total transmission

$T=|C/A|^2=100\%$

When $\rho_2 \ll \rho_1 \Rightarrow B=A.\ C=2$!!