Lecture 9-1 Dispersion

There are 2 loop holes in the derivations in previous classes.

$\boxed{\omega = vk}$ is not true universally

Example 1: Damped wave eqn

$$(\rho\partial_t^2 + \epsilon\partial_t)\varphi = T\varphi$$

damping ϵ

use $\varphi = e^{ikx}e^{i\omega t}$

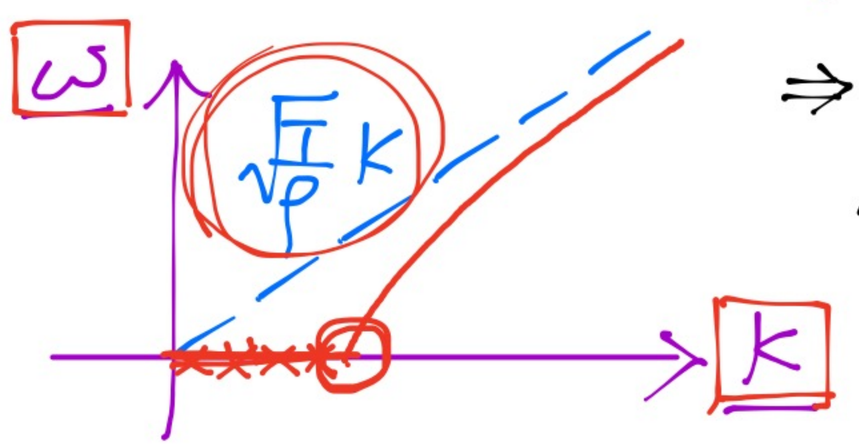
$$\Rightarrow -\rho\omega^2 + i\epsilon\omega + Tk^2 = 0$$

$$\Rightarrow \omega = \frac{1}{2\rho}\left(i\epsilon \pm \sqrt{4\rho T k^2 - 4\epsilon^2}\right)$$

$$= \frac{i\epsilon}{2\rho} \pm \sqrt{\frac{T}{\rho}k^2 - \frac{\epsilon^2}{\rho^2}}$$

$$= \pm\sqrt{\frac{T}{\rho}}\left(1 - \frac{\epsilon^2}{\rho T k^2}\right)^{1/2} k + \frac{i\epsilon}{2\rho}$$

$$\Rightarrow \varphi = e^{-\epsilon t/2\rho} e^{ikx} e^{i\omega t}, \quad \boxed{\omega = \sqrt{\frac{T}{\rho}}\left(k^2 - \frac{\epsilon^2}{\rho T}\right)^{1/2}}$$



$\Rightarrow \omega = \sqrt{\frac{T}{\rho}}k$ when $k \gg \frac{\epsilon}{\sqrt{\rho T}}$

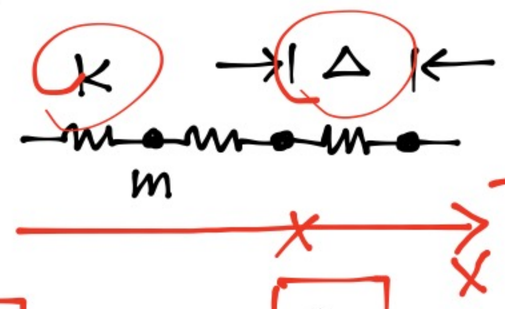
when $\boxed{k < \frac{\epsilon}{\sqrt{\rho T}}}$, all damping

Strange thing happens when $\boxed{k = \frac{\epsilon}{\sqrt{\rho T}}}$

double root $\Rightarrow$ critical damping

So what's the velocity of the wave?

Example 2:



See Lecture 4-1

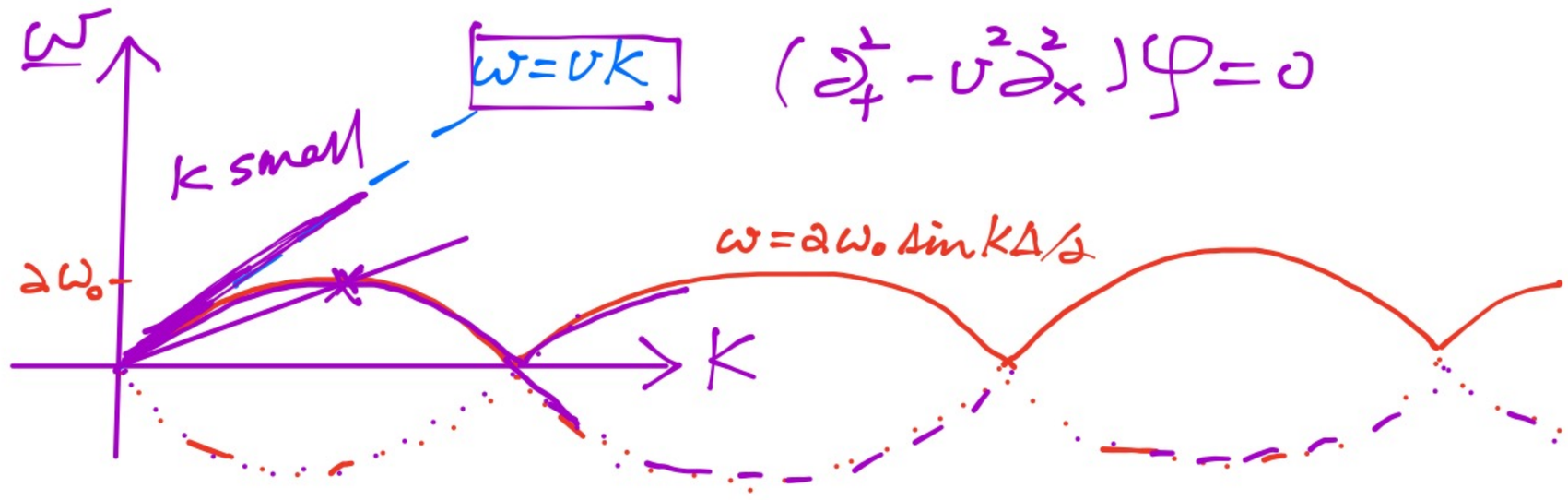
n-th oscillator

$$\boxed{\omega} = 2\omega_0 \sin \boxed{\epsilon/2} \qquad x_n = \sin n\epsilon\, e^{i\omega t} \qquad \boxed{\epsilon = \frac{m\pi}{N}}$$

$$\Rightarrow \varphi(x,t) = \sin \epsilon \underbrace{\boxed{\frac{x}{\Delta}}}_{k} e^{i\omega t} = \sin kx\, e^{i\omega t}$$

$$K = \frac{\epsilon}{\Delta}$$

$$\Rightarrow \omega = \boxed{2\omega_0} \sin \boxed{k\Delta/2} \xrightarrow[\text{small } k]{} \omega = \boxed{\omega_0 \Delta} k \equiv v k$$



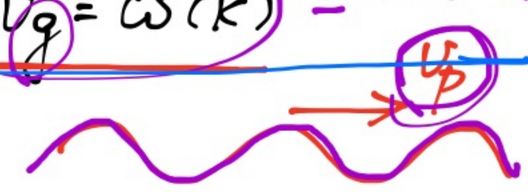
So in general, we have $\boxed{\omega = \omega(k)}$, called dispersion relation

what's exactly what's the propagation speed? $\boxed{e^{i(kx-\omega t)}}$

See $\boxed{\text{YouTube video}}$: phase velocity $v_p = \omega/k$ our old one

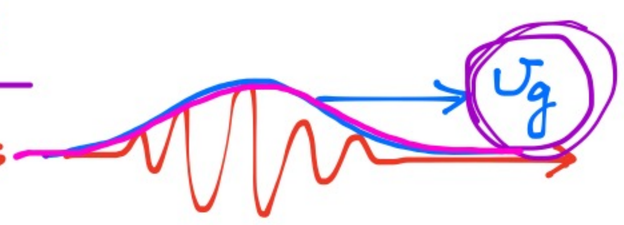
$\boxed{\text{group velocity } v_g = \omega'(k) = d\omega/dk}$

phase velocity : how sin & cos propagates

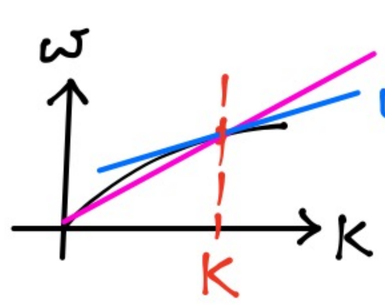


$\sin(kx-\omega t) = \sin k\boxed{(x - v_p t)}$

group velocity : how envelop propagates



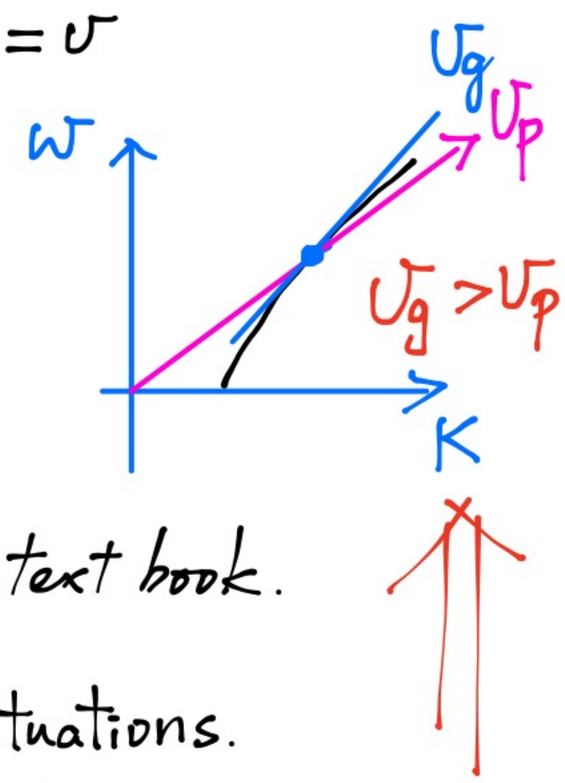
if $\omega(k)$ is linear $\omega = kv$. $\Rightarrow \boxed{v_p = v_g} = v$



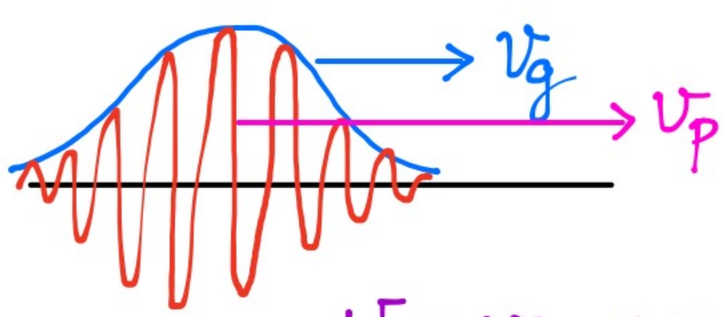
v_p is the ratio ω/k

v_g is the tangent $d\omega/dk$

$v_p > v_g$



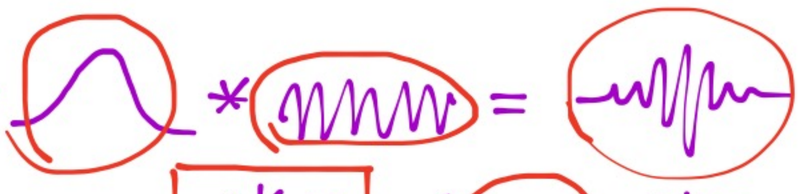
You can find 4~5 diff. proofs for v_g in the text book.



$\Rightarrow$ In typical situations. $v_g < v_p$, what we perceive can be slower than the phase velocity v_p.

$v_g > v_p$ *Also mean not impossible to propagate $> \sqrt{\frac{T}{\rho}}$?

$$= F(x) \times e^{ik_0 x}$$

Proof: $\varphi(x,t)\overset{=0}{}$ = 

$$= F(x)\boxed{e^{ik_0 x}} = \int \varphi(k) e^{ikx}\, dk$$

$$\Rightarrow F(x) = \int \varphi(k) e^{i(k-k_0)x}\, dk$$

$$\Rightarrow F(k) = e^{-ik_0 x}\varphi(k)$$

$$\Rightarrow \varphi(x,t) = \int \varphi(k) e^{ikx} e^{-i\omega t}\, dk \longleftarrow \text{general sol. } \omega(k)$$

$$= \int \varphi(k) e^{ik(x-vt)}\, dk \qquad \boxed{\text{if } \omega = vk \text{ is linear}}$$

$f(x-vt)$

$$= \varphi(x-vt, 0)$$

Now. $\omega(k) = \omega(k_0) + \omega'(k_0)(k-k_0) + \cdots$

$$= \omega_0 + v_g(k-k_0) + \cdots \qquad \omega_0 = v_p k_0$$

$$\varphi(x,t) \approx \int \varphi(k) e^{i(kx-\omega_0 t)} e^{-iv_g(k-k_0)t}\, dk \qquad \Delta k = k - k_0$$

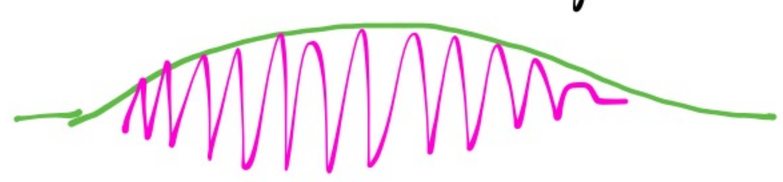
$$= \boxed{e^{i(k_0 x - \omega_0 t)}} \int \varphi(k) e^{i\Delta k x - iv_g \Delta k t}\, dk$$

$$= e^{i(k_0 x - \omega_0 t)} \underbrace{\int \varphi(k) e^{i(k-k_0)(x-v_g t)}\, dk}_{\boxed{F(x-v_g t)}}$$

$d\omega/dk$

$$\Rightarrow \varphi(x,t) = F(x-v_g t)\, e^{ik_0(x-v_p t)} \qquad \omega_0/k_0$$

Note the approximation we made, $\omega(k)$ only taken to leading order, $\Rightarrow F$ is a wide smooth function to ignore higher order terms.



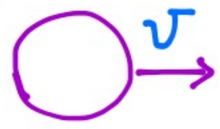
$\omega(k)$

Quantum Mechanics. $\omega(k) \longrightarrow E(p)$

$E = \hbar\omega, \; p = \hbar k$

Classical

particle



$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{d}{dp}\frac{p^2}{2m} = \frac{p}{m} = v_{cl}$$

Quantum

matterwave

$$v_p = \frac{\omega}{k} = \frac{E}{p} = \frac{\sqrt{m^2c^4 + \cancel{p^2c^2}}}{p} > v_{cl}$$

Lecture 9-2 Doppler Effect and shock waves

Among all miscellaneous wave phenomena in Chapters 6 & 7, these are the most interesting and useful.

Doppler effect: How we know the age of universe, and about big bang.

Doppler Radar: Meteorology, navigation, police, siren,

Laser cooling, tests of fundamental symmetry ...

Crazy physics: sonic boom. shock waves. light boom, Cherenkov radiation.

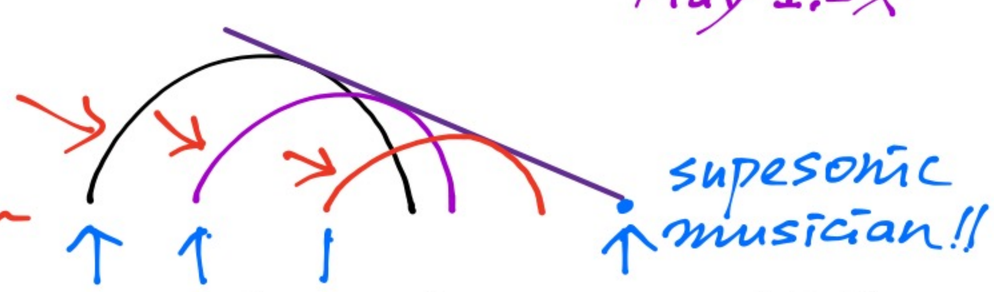
Bottom line:

boom!

Play I, IX

all music in a shot.

Laplacian



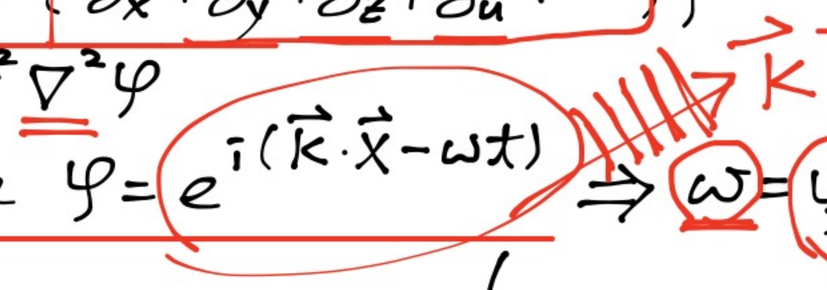
music played backwards.

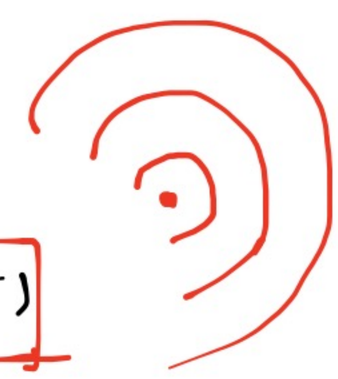
Wait, we have not learned waves in 2D!!

$$\Rightarrow \partial_t^2 \varphi = v^2 \boxed{(\partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_u^2 + \cdots)} \varphi \quad \text{wave eqn in D dimensions}$$

$$\equiv v^2 \nabla^2 \varphi$$

Plane wave $\varphi = e^{i(\vec{k}\cdot\vec{x} - \omega t)} \Rightarrow \omega = v_p \sqrt{k_x^2 + k_y^2 + \cdots}$





Spherial waves $\partial_t^2 \varphi = \frac{v^2}{r} \partial_r^2 (r\varphi) \Rightarrow \varphi = \frac{1}{r} \boxed{e^{i(k_r r - \omega t)}}$

because energy spreads out. weaker afar.

$$\Downarrow$$

$$\partial_t^2 (r\varphi) = v^2 \partial_r^2 (r\varphi)$$

$$r\varphi = e^{i(k - \omega t)}$$

Shock waves

What happen $v_a = \pm v_p$?

$v = v_p$ $\omega = 0$ hear nothing

$v = -v_p$ bass → soprano

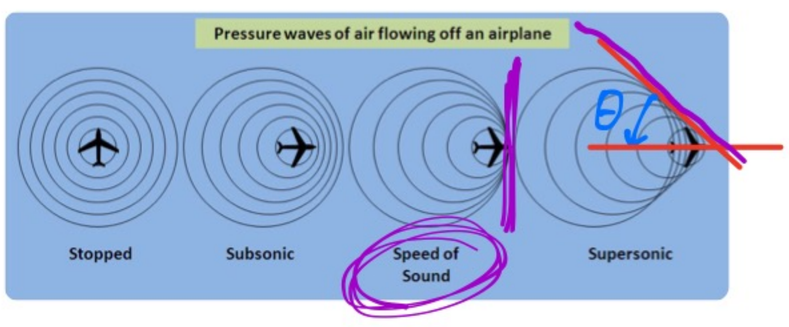
what happen $v_s = \pm v_p$

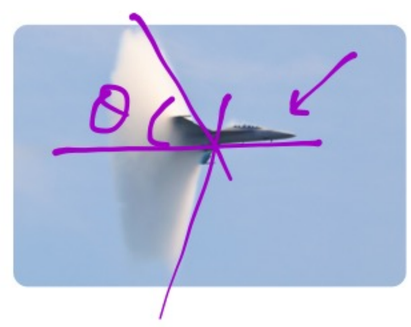
$v = +v_p$. $\omega \to \infty$ sonic / light boom

$v = -v_p$. $\omega = \omega_0/2$. soprano → bass

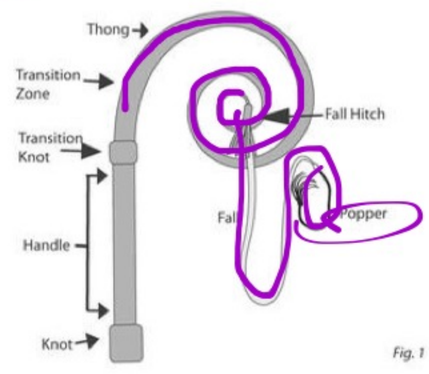
When source moves faster

It's not hard to show shock angle $\sin\theta = v_p / v_s \leq 1$

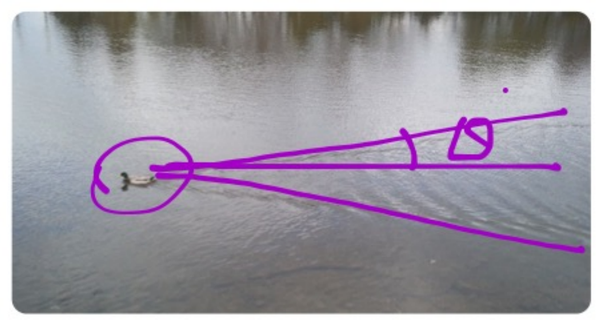




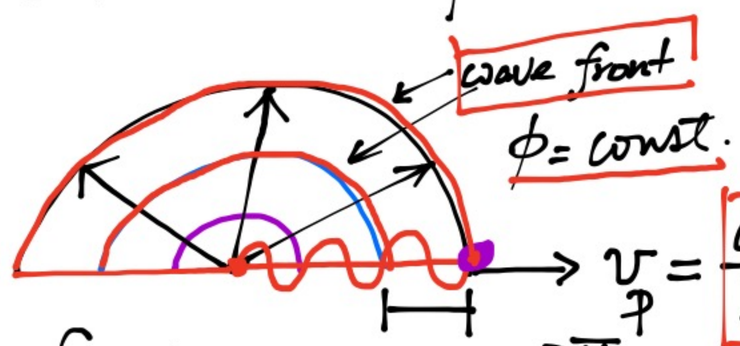
supersonic whip



Related phenomena: wake



Let's consider a point source



$v_p = \boxed{\dfrac{\omega_0}{K_0}}$

fixed musician

$\lambda_0 = \dfrac{2\pi}{K_0}$

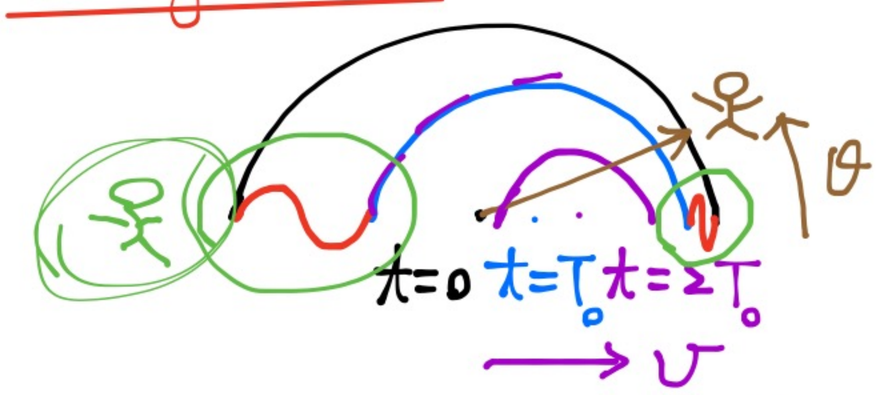
wavelength

- a full oscillation is released every period by the source

  audience gets $\omega = \text{rad/s}$

  $\omega_0 = 2\pi / T_0$

moving source



- Forward direction $\theta = 0$

$$\lambda = \lambda_0 - vT_0 \ll \lambda_0$$

$$\Rightarrow \omega = \frac{2\pi}{\lambda / v_p} = \omega_0 \boxed{\frac{\lambda_0}{\lambda}}$$

sound

$$\Rightarrow \omega = \omega_0 \frac{\lambda_0}{\lambda_0 - vT_0} = \frac{v_p}{v_p - v}\,\omega_0 > \omega_0$$

Blue shift $\omega > \omega_0$
(higher freq)

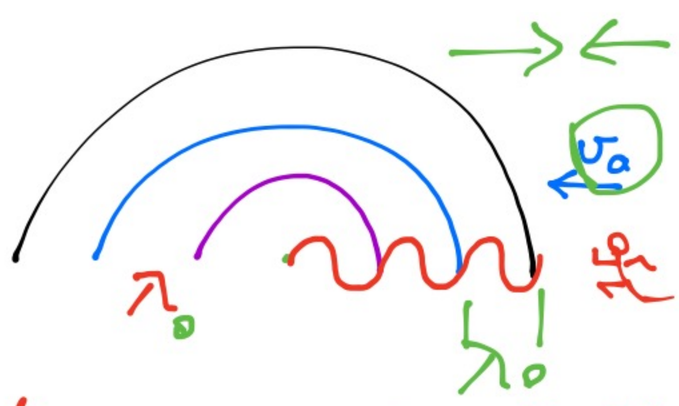
- Backward direction $\theta = \pi$

$$\Rightarrow \omega = \boxed{\frac{v_p}{v_p + v}}\,\omega_0 < \omega_0$$

red shift $\omega < \omega_0$

- In general: $\omega = \dfrac{v_p}{v_p - v\cos\theta}\,\omega_0$

Doppler effect for moving source



Audience runs away: $\omega = \omega_0\left(1 - \dfrac{v_a}{v_p}\right)$

red shift.

Audience running toward source

$$T = \frac{\lambda_0}{v_a + v_p} = T_0 \boxed{\left(1 + \frac{v_a}{v_p}\right)^{-1}} \qquad T_0 = \frac{\lambda_0}{v_p}$$

$$\omega = \frac{2\pi}{T} = \omega_0 \boxed{\left(1 + \frac{v_a}{v_p}\right)}$$

blue shift

red light looks green if you drive too fast.

What if both are running?

$$\omega = \omega_0 \frac{v_p - \hat{\ell}_{v_p} \cdot \vec{v}_a}{v_p - \hat{\ell}_{v_p} \cdot v_s}$$

audience speed

source speed.

direction of sound propagation