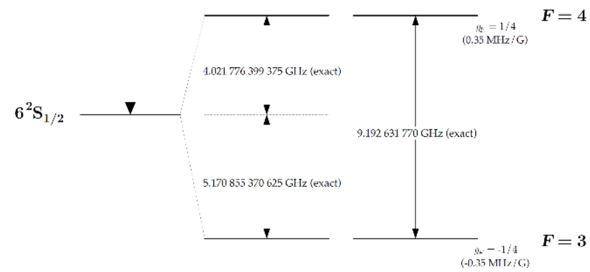
Physics 238: Atomic Physics

Fall Quarter 2021

Problem Set #1

Due: 12:20 pm, Tuesday, October 12. Please submit in class.

1. Structure of atomic ground state



The electronic ground state of cesium is labelled as $6^2S_{1/2}$, where "6" is the principal quantum number of the sole valence electron, ½ is the electron's spin *s*=1/2, and the super script 2 shows the spin degree of freedom of the electron.

The nuclear spin of a cesium atom is *i*=7/2, and this means the total angular momentum is $\boldsymbol{F} = \boldsymbol{s} + \boldsymbol{i}$ with angular quantum number $F = 3$ and 4. The splitting between the two states $\Delta E = h$ ×9192631770 Hz, see figure, adopted from Cs D Line data, is the primary frequency standard. The hyperfine splitting comes from the spin-spin interactions between the electron and the nucleus:

$$H = A\,\boldsymbol{s} \cdot \boldsymbol{i},$$

Show that the splitting is given by $\Delta E \equiv E_4 - E_3 = \left(i + \frac{1}{2}\right) A\hbar^2$.

(Hint: You may expand $\boldsymbol{F^2} = \boldsymbol{s^2} + \boldsymbol{2s} \cdot \boldsymbol{i} + \boldsymbol{i^2}$ and note that the eigenvalue of an angular momentum $\boldsymbol{L}$ satisfies $L^2|l >= l(l+1)\hbar^2|l >$. Evaluate the energy of the two hyperfine states $|F = s + i >$ and $|F = s - i >$.)

2. Magnetic dipole transition

In this problem we study the time evolution of a spin-1/2 atom in the presence of a static field in the z-direciton and an AC field in the radial direction $B = (B_1 \cos \omega t\, , B_1 \sin \omega t\, , B_0)$.

(1) Show that the Hamiltonian $H = -\boldsymbol{\mu} \cdot \boldsymbol{B}$ can be written in the matrix form as

$$H = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \Omega e^{-i\omega t} \\ \Omega e^{i\omega t} & -\omega_0 \end{pmatrix}$$

where $\boldsymbol{\mu} = -\frac{g}{2}\mu_B \boldsymbol{\sigma}$ is the magnetic moment, $g \approx 2$ is the electron g-factor, μ_B is the Bohr magneton and the angular momentum is given by the Pauli matrix $\boldsymbol{\sigma}$=$(\sigma_x, \sigma_y, \sigma_z)$. Determine the values of the Larmor frequency ω_0 and Rabi frequency Ω in terms of the magnetic field, g and μ_B.

(2) Here we introduce the spin wavefunction as $|\psi> = \begin{pmatrix}\psi_e \\ \psi_g\end{pmatrix}$ and the evolution of the wavefunction is given by the Schroedinger's equation $i\hbar\partial_t|\psi(t)> = H|\psi(t)>$. The general solution is $|\psi(t)> \equiv U(t)|\psi(0)>$, where the evolution operator is given by

$$U(t) = e^{-iHt/\hbar} = M\begin{bmatrix} e^{i\lambda_+ t} & 0 \\ 0 & e^{i\lambda_- t}\end{bmatrix}M^{-1}.$$

Show that $\lambda_\pm$ and M are given by the eigenvalues and eigenvetors of the Hamiltonian H. Derive the explicite forms of $\lambda_\pm$ and M.

Hint: $\lambda_\pm = \pm\frac{\Omega_R}{2}$, $\Omega_R = \sqrt{\Delta^2 + \Omega^2}$ is the generalized Rabi frequency and $M = RT$, where $R = \begin{pmatrix} e^{-\frac{i\omega t}{2}} & 0 \\ 0 & e^{\frac{i\omega t}{2}}\end{pmatrix}$ transforms the system to the rotating frame and $T = \frac{1}{\sqrt{2\Omega_R}}\begin{pmatrix} \sqrt{\Omega_R + \Delta} & \sqrt{\Omega_R - \Delta} \\ -\Omega/\sqrt{\Omega_R + \Delta} & \Omega/\sqrt{\Omega_R - \Delta}\end{pmatrix}$ transforms the system to the eigenstate basis.

(3) Given the initial condition $|\psi(0)> = \begin{pmatrix}0 \\ 1\end{pmatrix}$, show that the probability to find the particle in the excited state is given by the Rabi's formula:

$$|\psi_e(t)|^2 = \frac{\Omega^2}{\Omega_R^2}\sin^2\frac{\Omega_R t}{2}.$$

3. Radiative pulses in atom interferometry

Evolution operator $U(t)$ is used extensively to easily compute the quantum state after a sequence of pulses. Here we will explore applications in metrology and quantum information processing. Use the result of 2 and compute the following

(1) A θ-pulse is defined by a near resonant radiation $\omega \approx \omega_0$ with a pulse duration of $t = \frac{\theta}{\Omega}$. The associated evolution operator is given by U_θ. Determine the matrix form of $U_{\pi/2}$ and U_π in the basis of ground and excited states.

(2) Determine the free evolution operator $U(t)$ when the radiation is turned off ($B_1 = 0$) for a duration of time t ($B_0 = const.$ throughout the whole process).

(3) With the above operators, we can compute Ramsey spectroscopy following the following steps.

A: initialize atoms in the ground state $|\psi(0) >= \binom{0}{1}$

B: Apply a π-2 pulse. The wavefunction becomes $U_{\pi/2}|\psi(0) >$

C: Allow system to freely evolve for time t. The wavefunction becomes $U(t)U_{\pi/2}|\psi(0) >$.

D: Apply a second π-2 pulse.

What is the probability of the atoms in the excited state after the above steps.