

$$1. \vec{F}^2 = \vec{S}^2 + 2\vec{S} \cdot \vec{I} + \vec{I}^2$$

$$\vec{S} \cdot \vec{I} = \frac{1}{2} (\vec{F}^2 - \vec{S}^2 - \vec{I}^2)$$

$$H|FSI\rangle = A\vec{S} \cdot \vec{I}|FSI\rangle$$

$$= \frac{A}{2} (\vec{F}^2 - \vec{S}^2 - \vec{I}^2)|FSI\rangle$$

$$= \frac{A\hbar^2}{2} [F(F+1) - s(s+1) - i(i+1)]|FSI\rangle$$

$$\text{When } F=4, s = \frac{1}{2}, i = \frac{7}{2}$$

$$E_4 = \frac{A\hbar^2}{2} [4(4+1) - \frac{1}{2}(\frac{1}{2}+1) - \frac{7}{2}(\frac{7}{2}+1)]$$

$$= \frac{A\hbar^2}{2} (20 - \frac{3}{4} - \frac{63}{4}) = \frac{A\hbar^2}{2} \cdot \frac{7}{2} = \frac{7}{4} A\hbar^2$$

$$\text{When } F=3, s = \frac{1}{2}, i = \frac{7}{2}$$

$$E_3 = \frac{A\hbar^2}{2} [3(3+1) - \frac{1}{2}(\frac{1}{2}+1) - \frac{7}{2}(\frac{7}{2}+1)]$$

$$= \frac{A\hbar^2}{2} (12 - \frac{66}{4}) = \frac{A\hbar^2}{2} \cdot (-\frac{18}{4}) = -\frac{9}{4} A\hbar^2$$

$$E_4 - E_3 = 4A\hbar^2 \text{ equivalent to } \Delta E = (i + \frac{1}{2})A\hbar^2 \text{ when } i = \frac{7}{2}$$

$$2.1 \quad H = - \vec{\mu} \cdot \vec{B}$$

$$= \frac{g}{2} \mu_B \vec{\sigma} \cdot \vec{B}$$

$$= \frac{g}{2} \mu_B \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_1 \cos \omega t + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} B_1 \sin \omega t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B_0 \right\}$$

$$= \frac{g}{2} \mu_B \begin{pmatrix} B_0 & B_1 (\cos \omega t - i \sin \omega t) \\ B_1 (\cos \omega t + i \sin \omega t) & -B_0 \end{pmatrix}$$

$$= \frac{g}{2} \mu_B \begin{pmatrix} B_0 & B_1 e^{-i\omega t} \\ B_1 e^{i\omega t} & -B_0 \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \Omega e^{-i\omega t} \\ \Omega e^{i\omega t} & -\omega_0 \end{pmatrix}$$

$$\text{so } \omega_0 = \frac{g\mu_B B_0}{\hbar}, \quad \Omega = \frac{g\mu_B B_1}{\hbar}$$

$$2.2 \quad i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle$$

$$\rightarrow i\hbar \frac{d}{dt} (R|\psi\rangle) = H(R|\psi\rangle)$$

$$i\hbar (\dot{R}) |\psi\rangle + i\hbar R \frac{d}{dt} |\psi\rangle = HR |\psi\rangle$$

$$i\hbar R \frac{d}{dt} |\psi\rangle = (HR - i\hbar \dot{R}) |\psi\rangle$$

$$i\hbar \frac{d}{dt} |\psi\rangle = (R^{-1}HR - i\hbar R^{-1}\dot{R}) |\psi\rangle$$

$$\begin{aligned}
 R^{-1}HR &= \begin{pmatrix} e^{\frac{i\Omega t}{2}} & 0 \\ 0 & e^{-\frac{i\Omega t}{2}} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \Omega e^{-i\Omega t} \\ \Omega e^{i\Omega t} & -\omega_0 \end{pmatrix} \begin{pmatrix} e^{-\frac{i\Omega t}{2}} & 0 \\ 0 & e^{\frac{i\Omega t}{2}} \end{pmatrix} \\
 &= \frac{\hbar}{2} \begin{pmatrix} \omega_0 e^{\frac{i\Omega t}{2}} & \Omega e^{-\frac{i\Omega t}{2}} \\ \Omega e^{\frac{i\Omega t}{2}} & -\omega_0 e^{-\frac{i\Omega t}{2}} \end{pmatrix} \begin{pmatrix} e^{-\frac{i\Omega t}{2}} & 0 \\ 0 & e^{\frac{i\Omega t}{2}} \end{pmatrix} \\
 &= \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \Omega \\ \Omega & -\omega_0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 i\hbar R^{-1} \frac{d}{dt} R &= i\hbar \begin{pmatrix} e^{\frac{i\Omega t}{2}} & 0 \\ 0 & e^{-\frac{i\Omega t}{2}} \end{pmatrix} \frac{d}{dt} \begin{pmatrix} e^{-\frac{i\Omega t}{2}} & 0 \\ 0 & e^{\frac{i\Omega t}{2}} \end{pmatrix} \\
 &= i\hbar \begin{pmatrix} e^{\frac{i\Omega t}{2}} & 0 \\ 0 & e^{-\frac{i\Omega t}{2}} \end{pmatrix} \begin{pmatrix} -\frac{i\Omega}{2} e^{-\frac{i\Omega t}{2}} & 0 \\ 0 & \frac{i\Omega}{2} e^{\frac{i\Omega t}{2}} \end{pmatrix} \\
 &= i\hbar \begin{pmatrix} -\frac{i\Omega}{2} & 0 \\ 0 & \frac{i\Omega}{2} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix}
 \end{aligned}$$

$$R^{-1}HR - i\hbar R^{-1} \frac{d}{dt} R = \frac{\hbar}{2} \begin{pmatrix} \omega_0 - \Omega & \Omega \\ \Omega & -\omega_0 + \Omega \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{pmatrix}$$

$$\therefore i\hbar \frac{d}{dt} |\psi\rangle = \frac{\hbar}{2} \begin{pmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{pmatrix} |\psi\rangle \rightarrow i \frac{d}{dt} |\psi\rangle = \frac{1}{2} \begin{pmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{pmatrix} |\psi\rangle \dots \textcircled{D}$$

Next, we introduce the ansatz  $\psi = Ae^{-i\lambda t}$  into  $\textcircled{D}$

$$\lambda A = \frac{1}{2} \begin{pmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{pmatrix} A \dots \textcircled{2}$$

$$\begin{vmatrix} -\frac{\Delta}{2} - \lambda & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \frac{\Delta}{2} - \lambda \end{vmatrix} = -\left(\frac{\Delta}{2} + \lambda\right)\left(\frac{\Delta}{2} - \lambda\right) - \frac{\Omega^2}{4} = -\frac{\Delta^2}{4} + \lambda^2 - \frac{\Omega^2}{4} = 0$$

$$\lambda^2 = \frac{1}{4}(\Delta^2 + \Omega^2) \rightarrow \text{Eigenvalues } \lambda = \pm \frac{1}{2}\sqrt{\Delta^2 + \Omega^2}$$

$$\textcircled{2} \rightarrow \begin{pmatrix} -\Delta - 2\lambda_{\pm} & \Omega \\ \Omega & \Delta - 2\lambda_{\pm} \end{pmatrix} A_{\pm} = 0 \quad \text{where } A_{\pm} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{cases} (-\Delta - \Omega_R)\alpha + \Omega\beta = 0 \\ \Omega\alpha + (\Delta - \Omega_R)\beta = 0 \end{cases}$$

$$\text{when } \beta = -\frac{\Omega}{\Delta - \Omega_R} \alpha$$

$$\text{Normalization } |\alpha|^2 + |\beta|^2 = 1$$

$$\alpha^2 + \frac{\Omega^2}{(\Delta - \Omega_R)^2} \alpha^2 = \frac{\Delta^2 - 2\Omega_R\Delta + \Omega_R^2 + \Omega^2}{(\Delta - \Omega_R)^2} \alpha^2 = 1$$

$$\rightarrow \frac{2\Omega_R^2 - 2\Omega_R\Delta}{(\Delta - \Omega_R)^2} \alpha^2 = 1 \rightarrow \alpha = \pm \frac{\Delta - \Omega_R}{\sqrt{2\Omega_R^2 - 2\Omega_R\Delta}}$$

$$\alpha = \frac{1}{\sqrt{2\Omega_R}} \sqrt{\Omega_R - \Delta}, \quad \beta = \frac{1}{\sqrt{2\Omega_R}} \frac{\Omega}{\sqrt{\Omega_R - \Delta}}$$

$$\begin{cases} (-\Delta + \Omega_R)\alpha + \Omega\beta = 0 \\ \Omega\alpha + (\Delta + \Omega_R)\beta = 0 \end{cases}$$

$$\text{when } \beta = -\frac{\Omega}{\Delta + \Omega_R} \alpha$$

$$\alpha = \frac{1}{\sqrt{2\Omega_R}} \sqrt{\Omega_R + \Delta}, \quad \beta = -\frac{1}{\sqrt{2\Omega_R}} \frac{\Omega}{\sqrt{\Omega_R + \Delta}}$$

$$\therefore A_+ = \frac{1}{\sqrt{2\Omega_R}} \begin{pmatrix} \sqrt{\Omega_R + \Delta} \\ -\Omega/\sqrt{\Omega_R + \Delta} \end{pmatrix} \cdot A_- = \frac{1}{\sqrt{2\Omega_R}} \begin{pmatrix} \sqrt{\Omega_R - \Delta} \\ \Omega/\sqrt{\Omega_R - \Delta} \end{pmatrix}$$

The transformation matrix is

$$T = \frac{1}{\sqrt{2\Omega_R}} \begin{pmatrix} \sqrt{\Omega_R + \Delta} & \sqrt{\Omega_R - \Delta} \\ -\Omega/\sqrt{\Omega_R + \Delta} & \Omega/\sqrt{\Omega_R - \Delta} \end{pmatrix}$$

$$2.3 \quad |\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$$

$$\text{where } H = \frac{\hbar}{2} \begin{pmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{pmatrix} = \frac{\hbar}{2} (\Omega \sigma_x - \Delta \sigma_z) = \frac{\hbar}{2} (\vec{\sigma} \cdot \vec{\Omega})$$

$$\vec{\sigma} \cdot \vec{\Omega} = \vec{\sigma} \cdot \Omega_R \hat{\Omega} \quad \text{where } \hat{\Omega} = \left( \frac{\Omega}{\Omega_R}, 0, -\frac{\Delta}{\Omega_R} \right) \text{ unit vector}$$

since  $(\vec{\sigma} \cdot \hat{\Omega})^2 = I_2$ , we can rewrite  $e^{-iHt/\hbar}$  as

$$\begin{aligned} e^{-iHt/\hbar} &= \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{iHt}{\hbar} \right)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{i(\vec{\sigma} \cdot \vec{\Omega})t}{2} \right)^k \\ &\xrightarrow{\text{matrix exponential}} \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{i\Omega_R t}{2} \right)^k (\vec{\sigma} \cdot \hat{\Omega})^k \\ &= \sum_{n=0}^{\infty} \frac{(-i)^{2n}}{(2n)!} \left( \frac{\Omega_R t}{2} \right)^{2n} (\vec{\sigma} \cdot \hat{\Omega})^{2n} + \sum_{n=0}^{\infty} \frac{(-i)^{2n+1}}{(2n+1)!} \left( \frac{\Omega_R t}{2} \right)^{2n+1} (\vec{\sigma} \cdot \hat{\Omega})^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left( \frac{\Omega_R t}{2} \right)^{2n} I_2 - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \frac{\Omega_R t}{2} \right)^{2n+1} (\vec{\sigma} \cdot \hat{\Omega}) \end{aligned}$$

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From the Taylor series expansion, we have

$$\cos\left(\frac{\Omega_R t}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\Omega_R t}{2}\right)^{2n},$$

$$\sin\left(\frac{\Omega_R t}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\Omega_R t}{2}\right)^{2n+1}$$

Therefore, we get

$$e^{-iHt/\hbar} = \cos\left(\frac{\Omega_R t}{2}\right) I_2 - i \sin\left(\frac{\Omega_R t}{2}\right) (\vec{\sigma} \cdot \hat{\Omega})$$

$$= \cos\left(\frac{\Omega_R t}{2}\right) I_2 - i \sin\left(\frac{\Omega_R t}{2}\right) \left(\sigma_x \frac{\Omega}{\Omega_R} - \sigma_z \frac{\Delta}{\Omega_R}\right)$$

$$= \cos\left(\frac{\Omega_R t}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin\left(\frac{\Omega_R t}{2}\right) \frac{1}{\Omega_R} \begin{pmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{\Omega_R t}{2}\right) + \frac{i\Delta}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) & -\frac{i\Omega}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) \\ -\frac{i\Omega}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) & \cos\left(\frac{\Omega_R t}{2}\right) - \frac{i\Delta}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) \end{pmatrix}$$

$$\varphi(t) = e^{-iHt/\hbar} \varphi(0) = e^{-iHt/\hbar} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -i \frac{\Omega}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) \\ \cos\left(\frac{\Omega_R t}{2}\right) - i \frac{\Delta}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) \end{pmatrix}$$

$$\psi_e(t) = \langle e | \psi(t) \rangle = -i \frac{\Omega}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right)$$

There, the probability of finding the particle in the excited state is

$$\therefore |\psi_e(t)|^2 = \frac{\Omega^2}{\Omega_R^2} \sin^2\left(\frac{\Omega_R t}{2}\right)$$

3.1  $\Delta = \omega - \omega_0 = 0$ ,  $\Omega_R = \Omega$

$$U(t) = e^{-iHt/\hbar} = \begin{pmatrix} \cos\left(\frac{\Omega_R t}{2}\right) + \frac{i\Delta}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) & -\frac{i\Omega}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) \\ -\frac{i\Omega}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) & \cos\left(\frac{\Omega_R t}{2}\right) - \frac{i\Delta}{\Omega_R} \sin\left(\frac{\Omega_R t}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{\Omega_R}{2}t\right) & -i \sin\left(\frac{\Omega_R}{2}t\right) \\ -i \sin\left(\frac{\Omega_R}{2}t\right) & \cos\left(\frac{\Omega_R}{2}t\right) \end{pmatrix}$$

$$U_{\pi/2} = \begin{pmatrix} \cos\left(\frac{\Omega_R}{2} \cdot \frac{\pi}{2\Omega}\right) & -i \sin\left(\frac{\Omega_R}{2} \cdot \frac{\pi}{2\Omega}\right) \\ -i \sin\left(\frac{\Omega_R}{2} \cdot \frac{\pi}{2\Omega}\right) & \cos\left(\frac{\Omega_R}{2} \cdot \frac{\pi}{2\Omega}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -i \sin\left(\frac{\pi}{4}\right) \\ -i \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$\begin{aligned}
 U_{\pi} &= \begin{pmatrix} \cos\left(\frac{\Omega_R}{2} \cdot \frac{\pi}{\Omega}\right) & -i \sin\left(\frac{\Omega_R}{2} \cdot \frac{\pi}{\Omega}\right) \\ -i \sin\left(\frac{\Omega_R}{2} \cdot \frac{\pi}{\Omega}\right) & \cos\left(\frac{\Omega_R}{2} \cdot \frac{\pi}{\Omega}\right) \end{pmatrix} \\
 &= \begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & -i \sin\left(\frac{\pi}{2}\right) \\ -i \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}
 \end{aligned}$$

3.2 From  $\omega_0 = \frac{g\mu_B B_0}{\hbar}$ ,  $\Omega = \frac{g\mu_B B_1}{\hbar}$

when  $B_1 = 0$ ,  $\Omega = \Omega_R = 0$

$$\begin{aligned}
 U(t) &= \begin{pmatrix} \cos\left(\frac{\Omega_R}{2}t\right) & -i \sin\left(\frac{\Omega_R}{2}t\right) \\ -i \sin\left(\frac{\Omega_R}{2}t\right) & \cos\left(\frac{\Omega_R}{2}t\right) \end{pmatrix} \\
 &= \begin{pmatrix} \cos(0) & -i \sin(0) \\ -i \sin(0) & \cos(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
 \end{aligned}$$

3.3  $U_{\pi/2} U(t) U_{\pi/2} |\psi(0)\rangle$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\Omega_R}{2}t\right) & -i \sin\left(\frac{\Omega_R}{2}t\right) \\ -i \sin\left(\frac{\Omega_R}{2}t\right) & \cos\left(\frac{\Omega_R}{2}t\right) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\Omega_R}{2}t\right) & -i \sin\left(\frac{\Omega_R}{2}t\right) \\ -i \sin\left(\frac{\Omega_R}{2}t\right) & \cos\left(\frac{\Omega_R}{2}t\right) \end{pmatrix} \begin{pmatrix} -i \\ 1 \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} -i \cos\left(\frac{\Omega_R}{2}t\right) - i \sin\left(\frac{\Omega_R}{2}t\right) \\ \cos\left(\frac{\Omega_R}{2}t\right) - \sin\left(\frac{\Omega_R}{2}t\right) \end{pmatrix}
 \end{aligned}$$

$$= \frac{1}{2} \begin{pmatrix} -i \cos\left(\frac{\Omega_R t}{2}\right) - i \sin\left(\frac{\Omega_R t}{2}\right) - i \cos\left(\frac{\Omega_R t}{2}\right) + i \sin\left(\frac{\Omega_R t}{2}\right) \\ - \cos\left(\frac{\Omega_R t}{2}\right) - \sin\left(\frac{\Omega_R t}{2}\right) + \cos\left(\frac{\Omega_R t}{2}\right) - \sin\left(\frac{\Omega_R t}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -i \cos\left(\frac{\Omega_R t}{2}\right) \\ -\sin\left(\frac{\Omega_R t}{2}\right) \end{pmatrix}$$

$$\psi_e(t) = \langle e | \psi(t) \rangle = -i \cos\left(\frac{\Omega_R t}{2}\right)$$

$$\therefore |\psi_e(t)|^2 = \cos^2\left(\frac{\Omega_R t}{2}\right)$$

$$\text{when } \Omega_R = 0, |\psi_e(t)|^2 = 1$$

Alternatively, you could calculate

$$U(t) = M \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} M^{-1} \text{ where } M = RT$$

$$\text{and } R = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} \text{ and}$$

$$T = \frac{1}{\sqrt{2\Omega_R}} \begin{pmatrix} \sqrt{\Omega_R + \Delta} & \sqrt{\Omega_R - \Delta} \\ -\Omega_R / \sqrt{\Omega_R + \Delta} & \Omega_R / \sqrt{\Omega_R - \Delta} \end{pmatrix}$$