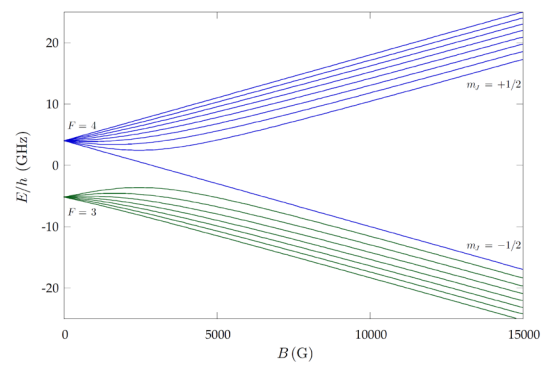
Physics 238: Atomic Physics
Fall Quarter 2021
Problem Set #2

Due: 12:20 pm, Thursday, October 19. Please submit in class.

1. **Zeeman shifts**

Here we continue our model in Problem 1 in Homework #1 to understand atomic structure. In the presence of magnetic field, the ground states of an alkali atom with one valence electron *s*=1/2 and nuclear spin *i* further split into magnetic level, which can be labeled as $|F, m_F; s = \frac{1}{2}, i >$, where $F = i \pm s$ is the total angular momentum quantum number, and $m_F = -F, -(F-1) \ldots F-1, F$ is the magnetic quantum number. As an example, the figure shows the splitting of a ground state cesium atoms with *s*=1/2, *i*=7/2.



To model the Zeeman effect, we include the magnetic dipolar interaction as

$$H = A\, \boldsymbol{s} \cdot \boldsymbol{i} - \boldsymbol{\mu} \cdot \boldsymbol{B},$$

where the term $\boldsymbol{s} \cdot \boldsymbol{i}$ is diagonal in the basis of $F = i \pm s$, and Zeeman splitting comes predominately from the magnetic moment of the valence electron $\boldsymbol{\mu} \approx \frac{g\mu_B}{\hbar}\boldsymbol{s}$ interacting with the magnetic field *B*. Here $g \approx 2$ is the electron g-factor, μ_B is the Bohr magneton, $\boldsymbol{s} = \hbar\boldsymbol{\sigma}$ is electron angular momentum, and $\boldsymbol{\sigma}$ is the Pauli matrix. As an example, $s_z|m_s >= \hbar\sigma_z|m_s >= \hbar m_s|m_s >$ and $m_s = \pm\frac{1}{2}$.

Assuming a weak magnetic field along the quantization axis (z-axis), we may expand the Zeeman shifts of the sublevels $|F, m_F; s = \frac{1}{2}, i >$ to leading order as

$$E(B) \equiv < H >= E_F + g_F \mu_B m_F B + O(B^2).$$

Show that effective g-factor $g_F = \frac{1}{i+1/2}$ and $-\frac{1}{i+1/2}$ for *F=i+s* and *i-s*, respectively. This result yields an important physical picture that $\frac{m_F}{i+1/2}$ can be viewed as the projection of the election spin in the magnetic field direction, namely, $<\boldsymbol{\mu} \cdot \boldsymbol{B} > \propto g_F \mu_B m_F B$. Compare your result with the figure for *i=7/2*.

2. Improved Ramsey interferometry

Given the rotating wave approximation we may write the wavefunction in the rotating frame as

$$i\partial_t \phi(t) = \frac{1}{2}\left(\Omega_{\mathrm{x}}\sigma_x + \Omega_{\mathrm{y}}\sigma_y - \Delta\sigma_z\right)\phi(t),$$

where Ω_{x} and Ω_{y} are the Rabi frequency of the cosine and sine components of the driving field, $\Delta = \omega - \omega_0$ is the frequency detuning of the laser ω relative to the atoms ω_0.

Consider the following improved pulse sequence:

- Start with atoms in the ground state $\phi(0) \equiv \begin{pmatrix}\phi_e(0)\\ \phi_g(0)\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$.
- Apply the first $\frac{\pi}{2}$ −pulse with only the Ω_{x} component
- Free evolution for time τ
- Apply the second $\frac{\pi}{2}$ −pulse with only the Ω_{y} component
- Measure the population in the excited state $P_e \equiv |\phi_e(\tau)|^2$

(1) Show that the excited population after the sequence is $P_e = \frac{1}{2}(1 - \sin\Delta\tau)$.
(Hint: you may use the results from Homework #1 Problem 2 or the Bloch vector. Both should give you the same answer, but the latter is a lot easier and intuitive.)

(2) The result $P_e = \frac{1}{2}(1 - \sin\Delta\tau)$ shows that when the laser frequency only slightly deviates from the atomic transition ω_0, the excited state population can sense the deviation with the highest sensitivity as $\frac{dP_e}{d\Delta} \approx -\frac{\tau}{2}$. Explain this result using the Bloch vector picture.

3. Optical Bloch equation

The simpliest way to incorporate spontaneous emission of atoms in the excited state is baesd on the density matrix, which in the rotating frame can be written as $\rho = |\phi >< \phi| = \begin{pmatrix} \rho_{22} & \rho_{12} \\ \rho_{21} & \rho_{11} \end{pmatrix}$. Using $\phi \equiv \begin{pmatrix} \phi_e \\ \phi_g \end{pmatrix}$, we get $\rho_{22} = P_e = |\phi_e|^2$, $\rho_{11} = |\phi_g|^2$, and $\rho_{12} = \rho_{21}^* = \phi_e^* \phi_g$. Particle conservation gives $\rho_{11} + \rho_{22} = 1$.

(1) Use the Hamiltonian $i\partial_t \phi(t) = \frac{1}{2}(\Omega_x \sigma_x + \Omega_y \sigma_y - \Delta \sigma_z)\phi(t)$ and prove the following equation of motion for the density matrix

$$\rho_{22}'(t) = -\rho_{12}'(t) = \frac{i\Omega}{2}(\rho_{21} - \rho_{12}) - \Gamma \rho_{22}$$

$$\rho_{12}'(t) = \rho_{21}'^*(t) = -\frac{i\Delta}{2}\rho_{12} + \frac{i\Omega}{2}(\rho_{22} - \rho_{11}) - \frac{\Gamma}{2}\rho_{12}$$

Where the terms in RED are artifically introduced to capture the decay of excited state with a rate constant Γ. Find an argument why the decay constant for ρ_{12} is $\Gamma/2$.

(2) Starting with an atom in the ground state $\phi_g(0) = 1$, plot the solution of $\rho_{22}(t)$ based on the above equation with zero detuning $\Delta = 0$ and laser intensity parameter $\frac{I}{I_s} = \frac{2\Omega^2}{\Gamma^2} = 2$, where I_s is called the saturation intensity.
(Remark: I_s is a very useful parameter to characterize the strength of radiation by an atom. For instance, $I_s = 1.12$ mW/cm^2 for the first electronic excited state of Cs atom.)

(3) Show that the excited population ρ_{22} in the above example approaches the following general result frequently used in quantum optics laboratory

$$\rho_{22} = \frac{1}{2}\frac{p}{1+p},$$

and the saturation parameter characterize how much an atom is saturated by light, defined as

$$p = \frac{I/I_s}{1 + I/I_s + 4\Delta^2/\Gamma^2}.$$

(Hint: You do not have to derive the general formula, just show that the evolution in (2) does approach the value predicted by the formula.)