Physics 238: Atomic Physics
Fall Quarter 2021
Problem Set #4
Due: 12:20 pm, Tuesday, November 9. Please submit in class.

1. **Bose-Einstein condensation (BEC)**
In the class we defined the condition for BEC as there are 1 or more particles in the ground state $N_0 \geq 1$, and the total particle number is $N = \sum_{i=0} N_i$, where $N_i = N_0 e^{-\frac{E_i}{kT}}$ is the population in the i-th lowest quantum state of the system, and the ground state energy is set to $E_0 = 0$. In a large box of volume $V = L^3$, we have $E_i = \frac{p_i^2}{2m}$, where m is the atomic mass. We can rewrite the sum as the integral as $N = N_0 \int_0 e^{-\frac{E}{kT}} \rho(E) dE$, where $\rho(E) = \frac{2\pi(2m)^{\frac{3}{2}}}{h^3} E^{\frac{1}{2}} V$ is the density of state in the box.
   A. Complete the integral and show that the BEC condition can be written as $N_0 = n\lambda_{dB}^3 \geq 1$, where $n = \frac{N}{V}$ is the total particle density and the expression defines the thermal de Broglie wavelength λ_{dB}. Show that BEC occurs when $k_B T \leq k_B T_c \equiv \frac{h^2 n^{\frac{2}{3}}}{2\pi m}$, where T_c is the BEC critical temperature.
   B. Typical BEC experiments adopt magnetic or optical traps which offer a conservative harmonic potential we model as $V(r) = \frac{1}{2} m\omega^2 r^2$ with ω the trap frequency. We will explore whether the BEC condition depends on whether the trap is a box or a harmonic trap.
   Given the eigen-energies of an atom in the harmonic trap is $E = \hbar\omega(n_x + n_y + n_z)$, where $n_i = 0,1,2 \ldots$ you can derive $\rho(E) = \frac{E^2}{2\hbar^3\omega^3}$. Show that $N_0 = N\left(\frac{\hbar\omega}{k_B T}\right)^3$ and BEC occurs when $k_B T \leq k_B T_c \equiv N^{\frac{1}{3}} \hbar\omega$.
   C. It appears that we have different expressions for the critical temperature, but they correspond to the same physical condition. Consider the BEC in the harmonic trap at the critical temperature $T_c = k_B^{-1} N^{\frac{1}{3}} \hbar\omega$, show that the atomic density at the trap center $n(r=0)$ satisfies the BEC condition derived in A, namely,

$$k_B T_c = \frac{h^2 n(0)^{\frac{2}{3}}}{2\pi m}.$$

   This result offers an important insight that BEC occurs at the center of the trap. Moreover the condensation is a local quantum phenomena insensitive to the global trap geometry even so the calculation indicates.)

   (Hint: A thermal gas in a harmonic trap is normally distributed with density $n(r) = n(0)e^{-r^2/2\sigma^2}$, where the root-mean-square size of the sample $\sigma = \sqrt{< r^2 >}$ follows equipatition theorem $\frac{1}{2} m\omega^2\sigma^2 = \frac{1}{2} k_B T$.)

2. Wavefunction of an interacting Bose-Einstein condensate.

We start with non-interacting bosonic particles prepared in the ground state of a potential well $V(\vec{x})$, its wavefunction can be obtained by the standard Schroedinger equation

$$\left[\frac{p^2}{2m} + V(\vec{x})\right]\psi(\vec{x}) = E_0\psi(\vec{x}).$$

The above is generalized for interacting bosons in the ground state by introducing the interaction term $U = gn$, which, for typical BECs, is dominated by two-body collision term with coupling constant g, $n(\vec{x}) = |\phi(\vec{x})|^2$ is the density of the sample. The "many-body" wavefunction $\phi(\vec{x})$ satisfies the mean-field Gross-Pitaevskii (GP) equation

$$\left[\frac{p^2}{2m} + V(\vec{x}) + g|\phi(\vec{x})|^2\right]\phi(\vec{x}) = \mu\phi(\vec{x})$$

and is normalized to the total particle number as $N = \int |\phi(\vec{x})|^2 d\vec{x}$. Here the constant μ here is called the chemical potential of the BEC.

A. Thomas-Fermi approximation

The GP equation has no general analytic solution, but a good approximation applies to most experiments where the solution is simple, that is, when the kinetic energy term $\frac{p^2}{2m}$ is negligible compared to the potential and interaction terms. Here the omission of the kinetic energy term, called the Thomas-Fermi approximation, is valid when the trap is macroscopic in size. Show that under this approximation we have

$$n(\vec{x}) = \begin{cases} \frac{1}{g}[\mu - V(\vec{x})] & V(x) < \mu \\ 0 & V(\vec{x}) > \mu \end{cases}$$

B. Density profile of a harmonically trapped BEC

Let's consider a generic case of N interacting atoms cooled to the ground state of a spherical harmonic trap $V(\vec{x}) = \frac{1}{2}kr^2$ with interaction strength g. Show that the density profile is

$$n(r) = \begin{cases} \frac{\mu}{g}\left(1 - \frac{r^2}{R^2}\right) & r < R \\ 0 & r \geq R \end{cases},$$

where R is the Thomas-Fermi radius of the BEC. Derive the chemical potential and the radius

$$\mu = \left(\frac{15}{8\pi}Ng\right)^{\frac{2}{5}}\left(\frac{k}{2}\right)^{\frac{3}{5}}$$

$$R = \left(\frac{15}{8\pi}Ng\right)^{\frac{1}{5}}\left(\frac{k}{2}\right)^{\frac{1}{10}}$$

C. Density profile of a BEC in a box

What is the density profile and chemical potential of a BEC with N atoms confined in a large square well potential with volume V?