# Physics 238: Atomic Physics
## Fall Quarter 2021
Problem Set #6

Due: 12:00 pm, Thursday, December 2. Please submit in class.

**1. Low energy excitations of a Bose-Einstein condensate**

A Bose-Einstein condensate (BEC) can be described by the Gross-Pitaevskii equation under mean-field approximation,

$$i\hbar\partial_t\psi(x,t) = \left(\frac{p^2}{2m} + V(x) + g|\psi(x,t)|^2\right)\psi(x,t)$$

For a time-independent wavefunction with chemical potential $\mu$ we have

$$\left(\frac{p^2}{2m} + V(x) + g|\psi(x,t)|^2\right)\psi(x,t) = \mu\psi(x,t)$$

Here $p = -i\hbar\partial_x$ and we consider the condensate is confined in a large box $V(x) = 0$ with a uniform density $n$. The ground state wavefunction is thus $\psi_0(x,t) = n^{\frac{1}{2}}e^{-i\mu_0 t/\hbar}$, and the chemical potential is $\mu_0 = gn$,

A. Here we consider low energy excited states of the system, assuming the BEC is weakly perturbed. The wavefunction can be expanded as $\psi = \psi_0 + \epsilon\psi_1$, where $\epsilon \ll 1$. Up to first order in $\epsilon$, show that $\psi_1$ satisfies

$$i\hbar\partial_t\psi_1 = \left(\frac{p^2}{2m} + 2gn\right)\psi_1 + g\psi_0^2\psi_1^*,$$

which represents 2 coupled linear differential equations for $\psi_1$ and $\psi_1^*$.

B. Apply the ansatz $\psi_1 = e^{-i\mu_0 t/\hbar}[ue^{i(kx-\omega t)} + ve^{-i(kx-\omega t)}]$, where $u$ and $v$ are constant amplitudes of the plane waves, and show that they satisfy

$$\begin{pmatrix} \frac{\hbar^2k^2}{2m} + \mu_0 - \hbar\omega & \mu_0 \\ \mu_0 & \frac{\hbar^2k^2}{2m} + \mu_0 + \hbar\omega \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0$$

C. For solutions with non-zero amplitudes $\mu, v \neq 0$ show that the frequency and wave number are linked by the following dispersion $\omega(k)$

$$\omega = \frac{1}{\hbar}\sqrt{\frac{\hbar^2k^2}{2m}\left(\frac{\hbar^2k^2}{2m} + 2\mu_0\right)}$$

Thus the sound speed for long wavelength excitation $k \to 0$, is $v = \lim_{k\to 0}\frac{\omega}{k} = \sqrt{\frac{\mu_0}{m}}$.

**2. Low energy excitations of a Bose-Einstein condensate (second quantization)**

In the second quantization form, we can write the energy of a bosonic system as

$$H = \sum_k \epsilon_k a_k^+ a_k + \frac{g}{2V} \sum_{k1+k2=k3+k4} a_{k1}^+ a_{k2}^+ a_{k3} a_{k4},$$

where $a_k^+$ and $a_k$ creates and annihilates a boson with momentum $k$ and they satisfy the bosonic commutation relation $[a_k, a_{k'}^+] = \delta_{kk'}$, $\epsilon_k = \frac{\hbar^2 k^2}{2m}$ is the energy of a bare atom with momentum $k$ and $\frac{g}{V}$ is the interaction energy of a pair of atoms.

The wavefunction of the system can be described as $|\psi >= |n_1, n_2 \ldots >$, where $n_i$ is the population in the $i-$th lowest single atom eigenstate.

A. To gain some insight about the Hamiltonian, we assume there are only two momentum states $k = \pm 1$ in the system and there are only exactly 2 atoms that can occupy these states. The wavefunction can be a linear superposition of $|n_{-1}, n_1 >= |2,0>, |1,1>$ and|0,2>, where $n_{\pm 1}$ is the population in the $k = \pm 1$ state. Express the Hamiltonian as a matrix in the basis of the 3 states.

B. Now we consider the system where $N_0$ atoms are in the lowest momentum state $|\psi_0 >= |N_0, 0,0, \ldots >$. Show that the energy of the system is $< H >= \frac{g}{2V} N_0(N_0 - 1) \equiv E_0$.

C. Now we consider $N_0$>>1 atoms in the zero momentum $k = 0$ state and few atoms $N_i \ll N_0$ in the finite momentum states. Total particle number is $N = \sum_i N_i$. Approximating $N_0 \pm 1 \approx N_0$, show that we can approximate the system energy as

$$H = E_0 + \sum_{k \neq 0} \epsilon_k a_k^+ a_k + \frac{gN_0}{2V} \sum_{k \neq 0} (2a_k^+ a_k + a_k^+ a_{-k}^+ + a_k\, a_{-k})$$

*Remark: This result can be compared to the perturbation in 1 A.*

D. The Hamiltonian mixes states with $k$ and $-k$. Show that the following Bogoliubov transformation:

$$a_k = u_k \alpha_k + v_k \alpha_{-k}^+$$
$$a_k^+ = u_k \alpha_k^+ + v_k \alpha_{-k}$$

Show that with suitable choice of the coefficients $u_k$ and $v_k$ can diagonalize the Hamiltonian as

$$H = const. + \sum_{k \neq 0} \varepsilon_k \alpha_k^+ \alpha_k,$$

where $\alpha_k^+$ and $\alpha_k$ creates and annihilates a bosonic quasi-particle with momentum $k$. Since the Hamiltonian is diagonal in their population $\alpha_k^+ \alpha_k$. These quasi-particles are effective long-lived free particles in the system, called phonons that carry sound waves and do not interact with each other.

*Remark: Compare your result to 1B and 1C.*