10/5/2021 Bloch vector and Bloch sphere (Foot 7.3)

In practice (NMR, QIP, metrology ...) many pulses are applied

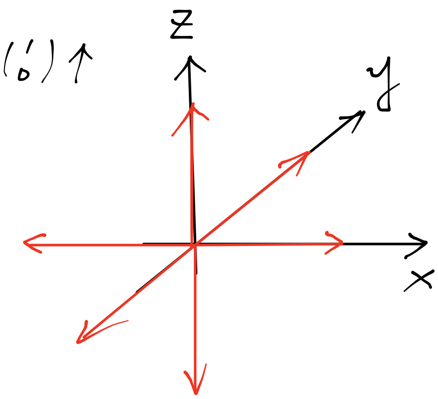
can we gain a better picture on the rotation of the spin?

An intuitive picture $\binom{0}{1} = |g\rangle \equiv \downarrow$ $\quad \binom{1}{0} = |e\rangle \equiv \uparrow$

$\hat{z}$ ground state $\binom{0}{1} \downarrow$ excited state $\binom{1}{0} \uparrow$

$\hat{x}$ $\binom{1}{1} \rightarrow$ $\binom{-1}{1} \leftarrow$

$\hat{y}$ $\binom{1}{i} \nearrow$ $\binom{-i}{1} \swarrow$



State Check degrees of freedom $\phi = \binom{\phi_e}{\phi_g}$

has 2 degrees of freedom $|\phi_e|^2 + |\phi_g|^2 = 1$

Spinor wavefunction $\longleftrightarrow$ vector in real 3D space.

Transform Isomorphism (Group theory)

Operation of a spinor $\binom{\phi_2}{\phi_1}$ $\approx$ Rotation of a 3D vector

SU(2) $\qquad$ SO(3)

special unitary group (2x2) $\qquad$ Special orthogonal group (3x3)

All $\hat{U}$ preserve probability $\qquad$ All rotations preserve length.

Formal definition of Bloch vector $\vec{b} \equiv \langle \vec{\sigma} \rangle = (\langle \sigma_x \rangle, \langle \sigma_y \rangle, \langle \sigma_z \rangle)$

Pauli matrices $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Check that $\binom{1}{0}$ gives $\vec{b} = (0.0.1)$

$\frac{1}{\sqrt{2}} \binom{1}{1}$ gives $\vec{b} = (1.0.0)$

$\frac{1}{\sqrt{2}} \binom{1}{i}$ gives $\vec{b} = (0.1.0)$

How does $\vec{b}$ evolve with time?

I. textbook (7.3.2)

Density matrix $\rho = |\,\rangle\langle\,| = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(c_1^*,\ c_2^*) = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_2 c_1^* & |c_2|^2 \end{pmatrix} \equiv \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$

$i\hbar \partial_t \rho = [H, \rho]$

$b = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle) = (\rho_{12}+\rho_{21},\ -i(\rho_{12}-\rho_{21}),\ \rho_{11}-\rho_{22})$

II. Heisenberg's picture $i\hbar\partial_t \langle F\rangle = [F, H]$

$\Rightarrow i\hbar \partial_t \vec{b} = [\langle\vec{\sigma}\rangle, H] = \langle[\vec{\sigma}, H]\rangle$

We assume $H = \hbar(h_0 \hat{I} + h_1 \hat{\sigma}_x + h_2 \hat{\sigma}_y + h_3 \hat{\sigma}_z) \equiv \hbar \sum h_i \sigma_i \quad \sigma_0 \equiv \hat{I}$

$$i\partial_t \vec{b} = \sum_{i=0}^{3} h_i \langle[\vec{\sigma}, \sigma_i]\rangle = \sum_{i=1}^{3} h_i \langle[\vec{\sigma}, \sigma_i]\rangle$$

$$= \sum_{ij} h_i \langle[\sigma_j, \sigma_i]\rangle \hat{e}_j$$

$$= 2i \sum_{ijk} \langle \epsilon_{jik} h_i \sigma_k\rangle \hat{e}_j$$

$$= 2i \sum_{ijk} \epsilon_{kij} h_i \langle\sigma_j\rangle \hat{e}_k$$

$[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\sigma_k$

$\epsilon_{ijk} = 1$ for (1.2.3) and all even permutations

$\epsilon_{ijk} = -1$ for all odd permutations

$\epsilon_{ijk} = 0$ for any repeated indexes

$\vec{A} \times \vec{B} = \epsilon_{ijk}\hat{e}_i A_j B_k$

$$\boxed{\Rightarrow \partial_t \vec{b} = 2\vec{h} \times \vec{b}}$$

Bloch vector eqn of motion.

Dynamics of Bloch vector $\approx$ angular rotation in mechanics

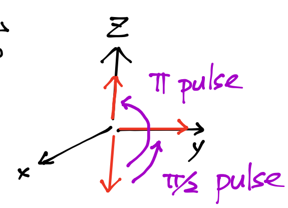
$\dot{\vec{r}} = \vec{\omega} \times \vec{r}$ or $\vec{\tau} = \dot{\vec{L}} = \mu \times B = -\gamma \vec{B} \times \vec{L}$

$\uparrow$ gyromagnetic ratio.

Example $i\partial_t \phi = -\frac{\Delta}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\phi + \frac{\Omega}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\phi$

$= \frac{1}{2}(\Omega, 0, -\Delta)\cdot(\sigma_x, \sigma_y, \sigma_z)\phi$

$\Rightarrow$ Bloch vector motion $\partial_t \vec{b} = (\Omega, 0, -\Delta) \times \vec{b}$



Resonant π-pulse

wavefunction $\phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

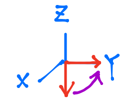
Bloch vector $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Compare with rigorous calculation

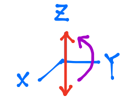
$\phi(t) = U(t)\phi(0)$ $\quad i\partial_t \phi = \left[-\frac{\Delta}{2}\sigma_z + \frac{\Omega_x}{2}\sigma_x + \frac{\Omega_y}{2}\sigma_y\right]\phi$, $\Omega_x - i\Omega_y \equiv \Omega \equiv |\Omega| e^{i\theta}$

resonant pulse $\Delta = 0$

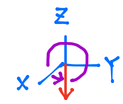
$\Rightarrow$ $\pi/2$ pulse $\quad t = \pi/2\Omega \quad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & ie^{i\theta} \\ -ie^{-i\theta} & 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} ie^{i\theta} \\ 1 \end{pmatrix}$



π pulse $\quad t = \pi/\Omega \quad U = \begin{pmatrix} 0 & ie^{i\theta} \\ -ie^{-i\theta} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} ie^{i\theta} \\ 0 \end{pmatrix}$



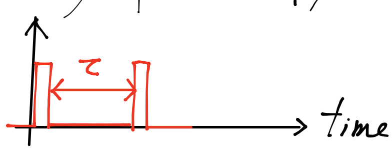
2π pulse $\quad t = 2\pi/\Omega \quad U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$



$$\text{Summary}: \; i\partial_t \phi = \frac{1}{2}(\underbrace{\Omega_x}_{\Omega_x \cos\omega t}\sigma_x + \underbrace{\Omega_y}_{\Omega_y \sin\omega t}\sigma_y - \underbrace{\Delta}_{\text{detuning},\ \Delta = \omega - \omega_0}\sigma_z)\phi \qquad \vec{b} \equiv \langle\vec{\sigma}\rangle \quad |\vec{b}| = 1$$

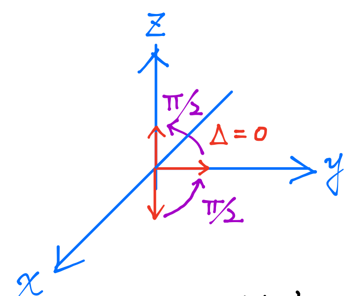
$\partial_t \vec{b} = 2\vec{h} \times \vec{b}$. rotation $\quad$ ular freq $2\vec{h} = (\Omega_x, \Omega_y, -\Delta)$

Exercise: Ramsey spectroscopy

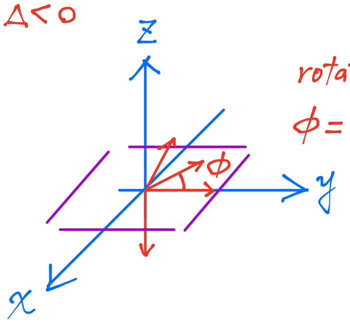


$\pi/2 \quad \pi/2$

$\Omega = \Omega_x$

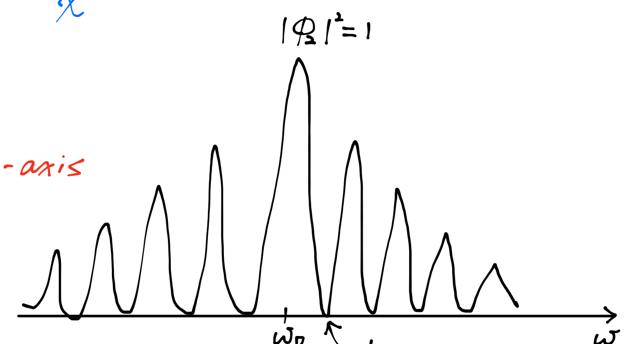


$\Delta < 0$



rotation along z-axis

$\phi = -\Delta\tau$

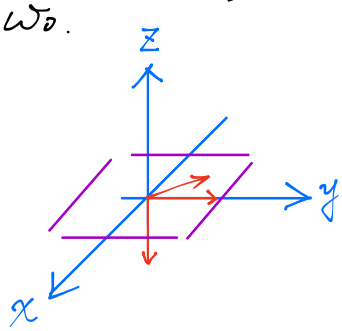


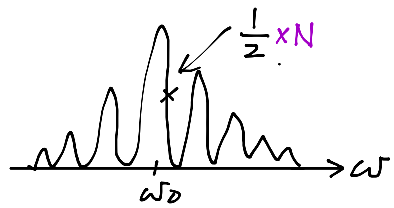
$\phi = -\Delta\tau = \pi$

$\Rightarrow \Delta = \pi/\tau$

$\Rightarrow \hbar\Delta\tau \; (= \Delta E \Delta t) = h/2$ quantum limit.

Practically we may use Ω_x for the 1st $\pi/2$ pulse & Ω_y for the 2nd one
So 1st rotation along $\hat{x}$ & 2nd along $\hat{y}$ to enhance sensitivity near $\omega = \omega_0$.

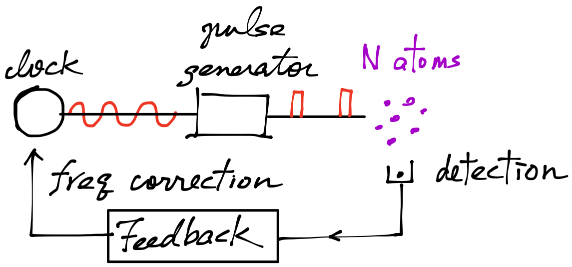




$$\text{Signal} = N \, \frac{1 - \sin(\omega - \omega_0)\tau}{2}$$

$$\delta S = \frac{N}{2} \tau \, \delta\omega \geq \frac{1}{2}\sqrt{N} \Rightarrow \delta\omega \geq \frac{1}{\tau\sqrt{N}}$$

Binomial distribution due to quantum projection



Ultimate freq sensitivity

$$\delta\omega = \frac{1}{\tau\sqrt{N}}$$

Lattice clock. $\tau = 1s$. $\omega = 2\pi \times 3 \times 10^{14}$

$N = 10^6 \Rightarrow \delta\omega = 2\pi \times 2 \times 10^{-4}\,Hz$

$\delta\omega / \omega = 5 \times 10^{-19}$