

Lecture 11 Gross-Pitaevskii Equation 2021P238 11/2/2021

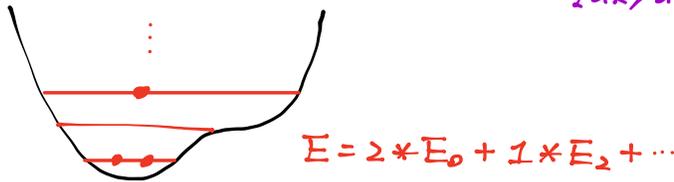
How do we describe N identical particles (Think about J.-C. model)

$| \rangle = | n_0, n_1, n_2 \dots \rangle$, no atoms in the lowest state and so on
in free space, we may use plane-wave momentum $\hbar k$ as quantum #.

$| \rangle = | n_{k_0}, n_{k_1}, n_{k_2} \dots \rangle$

In general n_i refers to n_i particles in the i -th eigenstate.

$| n, 0, 0, 0 \dots \rangle = \frac{1}{\sqrt{n!}} (a_0^\dagger)^n | 0 \rangle$, a_k^\dagger : bosonic creation of an atom
 $[a_k, a_{k'}^\dagger] = \delta_{kk'}$



If all particles are independent, $H = K + V$ (Kinetic + potential),

we get $H = \epsilon_0 n_0 + \epsilon_1 n_1 + \dots = \sum_i \epsilon_i a_i^\dagger a_i$

$$= \int dx \psi^\dagger(x,t) \left[\frac{p^2}{2m} + V \right] \psi(x,t) \quad \psi = \sum_i a_i \phi_i$$

Interaction $V = \frac{1}{2} \sum_{k \neq l} V(r_k - r_l)$

$$\psi^\dagger = \sum_i a_i^\dagger \phi_i^*$$

$$= \frac{1}{2} \int \psi^\dagger(r) \psi^\dagger(r') V(r,r') \psi(r') \psi(r) dr dr'$$

$$= \frac{1}{2} \sum_{\substack{k_1+k_2 \\ =k_3+k_4}} g_{k_1, k_2, k_3, k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}$$

For contact interaction $V = \frac{1}{2} g \sum_{\substack{k_1+k_2 \\ =k_3+k_4}} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}$

In a large box $H = \sum \epsilon_k a_k^\dagger a_k + \frac{g}{2} \sum_{\substack{k_1+k_2 \\ =k_3+k_4}} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}$
choose k as g #.

For instance: BEC with $N \gg 1$ atoms in the ground state.

$$|g\rangle = |N, 0, 0, \dots\rangle = \frac{1}{\sqrt{N!}} (a^\dagger)^N |0\rangle$$

$$\begin{aligned} H|g\rangle &= N\epsilon_0 |N, 0, 0, \dots\rangle + \frac{g}{2} \sum_k a_k^\dagger a_{-k}^\dagger \sqrt{N(N-1)} |N-2, 0, 0, \dots\rangle \\ &\approx N\epsilon_0 |N, 0, 0, \dots\rangle + \frac{g}{2} N(N-1) |N, 0, 0, \dots\rangle \end{aligned}$$

$$\Rightarrow \langle H \rangle \approx N\epsilon_0 + \frac{g}{2} N(N-1) \leftarrow \frac{1}{2} N(N-1) \text{ pairs of int. atoms}$$

Mean-field Gross-Pitaevskii Eqn

How do we describe N particles in the ground state?

$$H\psi(\vec{x}_1, \vec{x}_2, \dots) = E_0 \psi(\vec{x}_1, \vec{x}_2, \dots) \quad \vec{x}_i: \text{position of the } i\text{-th particle.}$$

Bose-Einstein condensation means all particles in the same state

$$\psi(\vec{x}_1, \vec{x}_2, \dots) = \phi(\vec{x}_1) \phi(\vec{x}_2) \dots \quad \phi(x): \text{ground state wavefunction}$$

Since all particles are indistinguishable, we introduce $\Psi(x) = \sqrt{N} \phi(x)$

s.t. $\int |\Psi(x)|^2 = N$, and the wavefunction satisfies

$$\bar{H}\Psi(x) = (\hat{P}_{2m}^2 + V + U)\Psi(x) \quad V: \text{potential energy } V(x)$$

$U: \text{interaction energy } U(n)$

$$(\hat{P}_{2m}^2 + V(x) + \underline{g n(x)})\Psi = \mu \Psi \quad \mu: \text{chemical potential} = \frac{\partial E}{\partial n}$$

mean-field potential

$$(\hat{P}_{2m}^2 + V(x) + g|\Psi(x)|^2)\Psi(x) = \mu \Psi(x)$$

This is the Gross-Pitaevskii Eqn.