

## HW1 Solution

1. (a)  $x'' + 4x' + 3x = 0$ ,  $x(0) = 1$ ,  $x'(0) = 1$

$$\text{Let } x = e^{\alpha t} \Rightarrow \alpha^2 + 4\alpha + 3 = 0 = (\alpha + 1)(\alpha + 3) \Rightarrow \alpha = -1, -3.$$

$$\Rightarrow x(t) = Ae^{-t} + Be^{-3t} \Rightarrow x'(t) = -Ae^{-t} - 3Be^{-3t}$$

$$\Rightarrow \left. \begin{array}{l} x(0) = A + B = 1 \\ x'(0) = -A - 3B = 1 \end{array} \right\} \Rightarrow \begin{array}{l} A = 2 \\ B = -1 \end{array}$$

$$\Rightarrow x(t) = 2e^{-t} - e^{-3t}$$

(b)  $x'' + 2x' + 5x = 0$ ,  $x(0) = 0$ ,  $x'(0) = -1$

$$\text{Let } x = e^{\alpha t} \Rightarrow \alpha^2 + 2\alpha + 5 = 0 \Rightarrow \alpha = -1 \pm \sqrt{-4} = -1 \pm 2i$$

$$\Rightarrow x(t) = Ae^{-t}e^{2it} + Be^{-t}e^{-2it}$$

$$\Rightarrow x'(t) = A(-1+2i)e^{-t}e^{2it} + B(-1-2i)e^{-t}e^{-2it}$$

$$\Rightarrow \left\{ \begin{array}{l} x(0) = A + B = 0 \\ x'(0) = A(-1+2i) - B(1+2i) = -1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} B = -A \\ A(-1+2i+1+2i) = 4iA = -1 \end{array} \right.$$

$$\Rightarrow A = \frac{i}{4}, B = -\frac{i}{4} \Rightarrow x(t) = \frac{i}{4}e^{-t}(e^{2it} - e^{-2it}) = -\frac{1}{2}e^{-t}\sin 2t$$

(c)  $x'' + 2x' + 5x = \sin t$ ,  $x(0) = x_0$ ,  $x'(0) = v_0$

$x = x_H + x_p$ . Homogeneous solution the same as (b).

particular solution  $x_p'' + 2x_p' + 5x_p = \sin t = \text{Im}[e^{it}]$

$$\text{Let } x_p = Ae^{it} \Rightarrow A(-1+2i+5)e^{it} = e^{it}$$

$$\Rightarrow A = \frac{1}{4+2i} = \frac{4-2i}{16+4} = \frac{2-i}{10}$$

$$\Rightarrow x_p = \text{Im}[Ae^{it}] = \frac{1}{5}\sin t - \frac{1}{10}\cos t$$

$$\Rightarrow x = \frac{1}{5}\sin t - \frac{1}{10}\cos t + Ae^{(-1+2i)t} + Be^{(-1-2i)t}$$

$$x' = \frac{1}{5}\cos t + \frac{1}{10}\sin t + A(-1+2i)e^{(-1+2i)t} + B(-1-2i)e^{(-1-2i)t}$$

$$\Rightarrow x(0) = \frac{-1}{10} = -\frac{1}{10} + A + B$$

$$x'(0) = 0 = \frac{1}{5} + A(-1+2i) + B(-1-2i) \Rightarrow \begin{array}{l} A+B=0 \\ 2i(A-B) = -\frac{1}{5} \end{array}$$

$$\Rightarrow A = \frac{i}{20}, B = -\frac{i}{20}$$

$$\begin{aligned} \Rightarrow x &= \frac{1}{5} \sin t - \frac{1}{10} \cos t + e^{-t} \frac{i}{20} (e^{2it} - e^{-2it}) \\ &= \frac{1}{5} \sin t - \frac{1}{10} \cos t - \frac{1}{20} e^{-t} 2 \sin 2t \\ &= \frac{1}{5} \sin t - \frac{1}{10} \cos t - \frac{1}{10} e^{-t} \sin 2t \end{aligned}$$

(d)  $x' + 2x = \cos t$ ,  $x(0) = 1$

Homogeneous solution:  $x_H' + 2x_H = 0$ .  $x_H = A e^{-2t}$

Particular solution:  $x_p' + 2x_p = e^{it}$

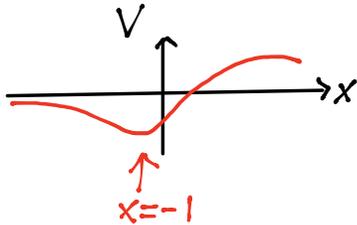
$$\text{Let } x = B e^{it} \Rightarrow B(i+2)e^{it} = e^{it} \Rightarrow B = \frac{1}{2+i} = \frac{2-i}{5}$$

$$\Rightarrow x_p = \text{Re} \left[ \frac{2-i}{5} e^{it} \right] = \frac{2}{5} \cos t + \frac{1}{5} \sin t$$

$$\Rightarrow x = \frac{2}{5} \cos t + \frac{1}{5} \sin t + A e^{-2t}$$

$$x(0) = \frac{2}{5} + A = 1 \Rightarrow A = \frac{3}{5} \Rightarrow x = \frac{2}{5} \cos t + \frac{1}{5} \sin t + \frac{3}{5} e^{-2t}$$

(e) First get an idea how  $V(x)$  looks like.



Determine the minimum

$$V' = \frac{1}{x^2+3} - \frac{(x-1)2x}{(x^2+3)^2} = 0$$

$$\Rightarrow x^2+3 = 2x(x-1)$$

$$\Rightarrow x^2 - 2x - 3 = (x-3)(x+1) = 0$$

$$\Rightarrow \text{minimum @ } x = -1$$

Near the minimum, we can expand  $x$  near  $x = -1$ :

$$V(x) \approx V(-1) + V'(-1)(x+1) + \frac{1}{2} V''(-1)(x+1)^2 + \dots$$

$$= -\frac{1}{8} + \frac{1}{2} \frac{1}{(x^2+3)^3} 2(x^3 - 3x^2 - 9x + 3) \Big|_{x=-1} (x+1)^2$$

$$= -\frac{1}{8} + \frac{1}{8} (x+1)^2$$

$$\Rightarrow x''(t) = F = -\frac{d}{dx} V(x) \approx -\frac{1}{4} (x+1). \quad \text{define } \chi = x+1$$

$$\Rightarrow \chi'' + \frac{1}{4} \chi = 0 \Rightarrow \chi(t) = \chi - 1 = A \sin \frac{t}{2} + B \cos \frac{t}{2} - 1.$$

$$(f) f(x, y) = x^{3/2}(x+4y)^{1/2} - (x+y)^2$$

$$\text{Consider } y/x = \epsilon \ll 1, f = x^2(1+4\epsilon)^{1/2} - x^2(1+\epsilon)^2$$

$$\text{Zeroth order: } \epsilon = 0, f = x^2 - x^2 = 0$$

$$\text{1st order: } f = x^2(1+2\epsilon) - x^2(1+2\epsilon) = 0$$

$$\begin{aligned} \text{2nd order: } f &= x^2(1+2\epsilon + \frac{1}{2} \frac{1}{2} (-\frac{1}{2})(4\epsilon)^2) - \\ & x^2(1+2\epsilon + \frac{1}{2} \cdot 2 \cdot 1 \epsilon^2) = x^2(-2\epsilon^2 - \epsilon^2) \\ &= -3x^2\epsilon^2 = -3x^2y^2/x^2 = -3y^2 \end{aligned}$$

$$\Rightarrow \lim_{x \gg y} f \approx -3y^2$$

$$2. (a) \text{ Given } x'' + 2\omega_0 x' + \omega_0^2 x = 0$$

$$\text{Let } x = e^{\alpha t} \Rightarrow \alpha^2 + 2\omega_0 \alpha + \omega_0^2 = (\alpha + \omega_0)^2 = 0 \Rightarrow \alpha = -\omega_0$$

$$\Rightarrow x = A e^{-\omega_0 t}, x' = -A\omega_0 e^{-\omega_0 t}$$

$$\begin{aligned} \Rightarrow \text{Total energy } E &= \frac{1}{2} m \omega_0^2 x^2 + \frac{1}{2} m x'^2 \\ &= \frac{1}{2} m \omega_0^2 A^2 e^{-2\omega_0 t} + \frac{1}{2} m \omega_0^2 A^2 e^{-2\omega_0 t} \end{aligned}$$

$$\Rightarrow \text{Decay time scale: } E(\tau)/E(0) = \frac{1}{2} = e^{-2\omega_0 \tau} \Rightarrow \tau = 1/2\omega_0$$

\* The other solution  $x = A t e^{-\omega_0 t}$  for critical damping gives the same time scale.

$$(b) \text{ Consider } \gamma^* \gg 2\omega_0$$

$$\alpha^2 + \gamma^* \alpha + \omega_0^2 = 0 \Rightarrow$$

$$\alpha = \frac{1}{2} [-\gamma^* \pm \sqrt{\gamma^{*2} - 4\omega_0^2}]$$

$$= \frac{1}{2} [-\gamma^* \pm \gamma^* (1 - 4\omega_0^2/\gamma^{*2})^{1/2}]$$

$$\approx \frac{1}{2} \gamma^* [-1 \pm (1 - 2\omega_0^2/\gamma^{*2})] = -\frac{\omega_0^2}{\gamma^*} \text{ or } -\gamma^*$$

$$\Rightarrow x = A e^{-(\omega_0^2/\gamma^*)t} + B e^{-\gamma^* t}$$

$$x' = -\frac{\omega_0^2}{\gamma^*} A e^{-(\omega_0^2/\gamma^*)t} - \gamma^* B e^{-\gamma^* t}$$

Assume @  $t=0$   $x=x_0$   $v=v_0 \Rightarrow \begin{cases} A+B=x_0 \\ \frac{\omega_0^2}{\gamma^*} A + \gamma^* B = -v_0 \end{cases}$

$$\Rightarrow \begin{cases} A = \frac{\gamma^* x_0 + v_0}{\gamma^* - \omega_0^2/\gamma^*} \approx x_0 + \frac{1}{\gamma^*} v_0 \\ B = x_0 - A \approx -\frac{1}{\gamma^*} v_0 \end{cases}$$

$$x(0)=1, x'(0)=0 \Rightarrow A=1, B=0$$

only the slow-decaying term is left.

$$\Rightarrow x = e^{-(\omega_0^2/\gamma^*)t}$$

$$\Rightarrow x' = -\frac{\omega_0^2}{\gamma^*} e^{-(\omega_0^2/\gamma^*)t}$$

$$V = \frac{1}{2} m \omega_0^2 e^{-2(\omega_0^2/\gamma^*)t} = V(0) e^{-2(\omega_0^2/\gamma^*)t}$$

$$E_K = \frac{1}{2} m x'^2 = E_K(0) e^{-2(\omega_0^2/\gamma^*)t}$$

$\Rightarrow$  Both energies decay with a rate of  $\mu = 2\omega_0^2/\gamma^*$ .

Note that when damping  $\gamma^* \rightarrow \infty$ , decay rate  $\mu \rightarrow 0$ , which means no energy loss??

(c) Physics picture



no damping  
 $\Rightarrow$  marble oscillates forever  
 $\Rightarrow$  no loss of energy



moderate damping  
 $\Rightarrow$  marble damps to center at the rate of  $\sim \gamma$



very strong damping  
 $\Rightarrow$  marble can hardly move in honey.  $\mu \sim \frac{2\omega_0^2}{\gamma^2}$

$$3. \quad x'' + \omega_0^2 x = f \cos \omega t$$

Steady state  $\Rightarrow$  particular solution.

$$\text{Consider } x'' + \omega_0^2 x = f e^{i\omega t}$$

$$x = A e^{i\omega t} \Rightarrow A(-\omega^2 + \omega_0^2) = f \Rightarrow A = \frac{f}{\omega_0^2 - \omega^2}$$

$$\text{Let } \omega = (1 - \epsilon)\omega_0 \Rightarrow \omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \\ \approx 2\epsilon\omega_0^2$$

$$\Rightarrow A = \frac{f}{2\epsilon\omega_0^2} \quad \& \quad x = \frac{f}{2\epsilon\omega_0^2} \cos \omega t \quad x' = \frac{-\omega f}{2\epsilon\omega_0^2} \sin \omega t$$

$$\Rightarrow \mathcal{E} = \frac{m}{2} x'^2 + \frac{m}{2} \omega_0^2 x^2 = \frac{m}{2} \frac{\omega^2 f^2}{4\epsilon\omega_0^4} \sin^2 \omega t + \frac{m}{2} \frac{\omega_0^2 f^2}{4\epsilon\omega_0^4} \cos^2 \omega t$$

$$\approx \frac{mf^2}{8\epsilon\omega_0^2}$$

$$(b) \quad x = A \sin \omega_0 t + B \cos \omega_0 t + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t$$

$$x' = A\omega_0 \cos \omega_0 t - B\omega_0 \sin \omega_0 t - \frac{f\omega}{\omega_0^2 - \omega^2} \sin \omega t$$

$$x(0) = B + \frac{f}{\omega_0^2 - \omega^2} = 0 \Rightarrow \begin{cases} A = 0 \\ B = -\frac{f}{\omega_0^2 - \omega^2} \end{cases}$$

$$x'(0) = A\omega_0 = 0$$

$$\Rightarrow x = \frac{f}{\omega_0^2 - \omega^2} (\cos \omega t - \cos \omega_0 t)$$

$$\approx \frac{-f}{2\epsilon\omega_0^2} 2 \sin \frac{\omega + \omega_0}{2} t \sin \frac{\omega - \omega_0}{2} t$$

$$= \frac{f}{\cancel{\epsilon\omega_0^2}} \sin \omega_0 t \quad \frac{\omega_0}{2} \cancel{\epsilon} t = \frac{f}{2\omega_0} t \sin \omega_0 t$$

$$\Rightarrow x' = \frac{f}{2\omega_0} (\sin \omega_0 t + \omega_0 t \cos \omega_0 t)$$

$$\Rightarrow \mathcal{E} = \frac{m}{2} x'^2 + \frac{m}{2} \omega_0^2 x^2 = \frac{mf^2}{8} \left[ t^2 + \frac{1}{\omega_0^2} \sin^2 \omega_0 t + \frac{t}{\omega_0} \sin 2\omega_0 t \right]$$

Average over one cycle:

$$\langle Z \rangle = \frac{1}{8} m f^2 t^2 + \frac{m f^2}{16 \omega_0^2} \approx \frac{m}{8} f^2 t^2$$

$$(c) Z_{\max} = \frac{m f^2}{8 \omega_0^2 \epsilon^2} = \frac{1}{8} m f^2 t^2$$

$$\Rightarrow \text{time} = t = \frac{1}{\omega_0 \epsilon} \text{ diverges as } \epsilon \rightarrow 0.$$

4. Chandler follows the pendulum eqn of motion:

$$m x'' + m \gamma x' + \frac{m g}{L} x = 0$$

$$(a) \Rightarrow \omega_0 = \sqrt{10} / s = 2\pi \times 0.5 \text{ Hz}, Q = 316$$

When the ceiling moves by  $\epsilon(t)$ , chandelier effectively displaces by  $x - \epsilon$ . Eqn becomes

$$x'' + \gamma x' + \omega_0^2 (x - \epsilon) = 0$$

$$\Rightarrow x'' + \gamma x' + \omega_0^2 x = \omega_0^2 \epsilon(t)$$

$$\text{particular solution: } A = \frac{\omega_0^2 \epsilon}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} = \frac{\omega_0 \epsilon}{\gamma}$$

$$\Rightarrow \epsilon = A \gamma / \omega_0 = 0.03 * 0.01 / \sqrt{10} = 10^{-4} \text{ m}$$

(b) It only works within a narrow band near  $\omega_0$ . So not a great seismometer.

