

HW2 Solution 2021P143b Cheng Chen

1. (a)  $\begin{pmatrix} A-A & \vec{U} \\ A & A \end{pmatrix} \vec{U} = \lambda \vec{U} \Rightarrow \begin{vmatrix} A-\lambda & -A \\ A & A-\lambda \end{vmatrix} = (A-\lambda)^2 + A^2 = 0$

 $\Rightarrow \lambda^2 - 2A\lambda + 2A^2 = 0 \Rightarrow \lambda = A \pm \sqrt{A^2 + A^2} = A(1 \pm i) \leftarrow \text{eigenvalues}$ 
 $\lambda_{\pm} = A(1 \pm i) \Rightarrow A \begin{pmatrix} 1-i & -1 \\ 1 & 1+i \end{pmatrix} \vec{U}_{\pm} = A(1 \pm i) \vec{U}_{\pm}$ 
 $\Rightarrow \begin{pmatrix} 1-i & -1 \\ 1 & 1+i \end{pmatrix} \vec{U}_{\pm} = 0, \text{ let } \vec{U}_{\pm} = \begin{pmatrix} x_{\pm} \\ y_{\pm} \end{pmatrix} \downarrow \text{eigenvectors}$ 
 $\Rightarrow \begin{pmatrix} \mp i & -1 \\ 1 & \mp i \end{pmatrix} \begin{pmatrix} x_{\pm} \\ y_{\pm} \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \mp i x_{\pm} - y_{\pm} \\ x_{\pm} + \mp i y_{\pm} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{U}_{\pm} = \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$

(b)  $\begin{pmatrix} A-A & \vec{V} \\ -A & 0-A \\ A & -A \end{pmatrix} \vec{V} = \lambda \vec{V} \Rightarrow \begin{vmatrix} A-\lambda & -A & A \\ -A & -\lambda & -A \\ A & -A & A-\lambda \end{vmatrix} = -\lambda(A-\lambda)^2 + A^3 + A^3 + \lambda A^2 - A^2(A-\lambda) - A^2(A-\lambda) = 0$

$\Rightarrow \lambda(\lambda-A)^2 - 2A^2(\lambda-A) - A^2\lambda - 2A^3 = \lambda(\lambda^2 - 2A\lambda - 2A^2) = 0$

$\Rightarrow \lambda = 0, (1 \pm \sqrt{3})A \equiv \lambda_0, \lambda_{\pm} \leftarrow \text{eigenvalues}$

$\lambda_0 = 0 \Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} \vec{U}_0 = 0 \Rightarrow \vec{U}_0 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda_{\pm} = 1 \pm \sqrt{3} \Rightarrow \begin{pmatrix} \mp \sqrt{3} & -1 & 1 \\ -1 & \mp \sqrt{3} & -1 \\ 1 & -1 & \mp \sqrt{3} \end{pmatrix} \vec{U}_{\pm} = 0. \quad \downarrow \text{eigenvectors.}$

$\text{Let } \vec{U}_{\pm} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} \mp \sqrt{3}x - y + z = 0 \\ x - y - \mp \sqrt{3}z = 0 \end{pmatrix} \Rightarrow \vec{U}_{\pm} = \begin{pmatrix} 1 \\ 1 \mp \sqrt{3} \\ 1 \end{pmatrix}$

(c)  $V(x, y) = x^2 + y^2 - xy - 3x. \quad \frac{\partial}{\partial x} V = 2x - y - 3 = 0 \Rightarrow x_0 = 2, y_0 = 1$   
 $\frac{\partial}{\partial y} V = 2y - x = 0$

Introduce  $u = x - 2, v = y - 1 \Rightarrow V$  has a minimum  $\Leftrightarrow (u, v) = 0$ .

$\Rightarrow V = (u+2)^2 + (v+1)^2 - (u+2)(v+1) - 3(u+2)$

$= u^2 + v^2 + 4u + 2v + 5 - uv - 2u - v - 3u - 6 - 2$

$= u^2 + v^2 - uv - 3$

$\vec{x} = m \vec{x}'' = m \begin{pmatrix} u \\ v \end{pmatrix}'' = \begin{pmatrix} -\partial_u V \\ -\partial_v V \end{pmatrix} = \begin{pmatrix} -2u+v \\ -2v+u \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$\Rightarrow \vec{x}'' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} \quad \text{let } \vec{x}(t) = \vec{A} e^{\alpha t}$

$\Rightarrow \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{A} = \alpha^2 \vec{A} \Rightarrow \begin{vmatrix} -2-\alpha^2 & 1 \\ 1 & -2-\alpha^2 \end{vmatrix} = (\alpha^2 + 2)^2 - 1 = 0$

$$\Rightarrow \alpha^2 + 2 = \pm 1 \Rightarrow \alpha^2 = -1, -3, \Rightarrow \alpha = \pm i, \pm \sqrt{3}i$$

$$\text{For } \alpha = \pm i, \vec{x} = \vec{A} e^{it} + \vec{B} e^{-it} = \vec{C} \cos t + \vec{D} \sin t$$

$$\alpha = \pm \sqrt{3}i, \vec{x} = \vec{E} \cos \sqrt{3}t + \vec{F} \sin \sqrt{3}t.$$

$\Rightarrow$  Eigenfrequencies are 1 and  $\sqrt{3}$ .

$$(d) \text{ Assume } w(x, y) = x^2 + y^2 - xy$$

Since  $w(x, y)$  has the minimum  $\nabla w(x, y) = 0$ .

$V = e^{\omega t}$  also has the minimum  $\nabla V(x, y) = 0$

$$\vec{F} = \vec{x}'' = \begin{pmatrix} -\partial_x V \\ -\partial_y V \end{pmatrix} = -\begin{pmatrix} V \partial_x \omega \\ V \partial_y \omega \end{pmatrix} = -\begin{pmatrix} \partial_x \omega \\ \partial_y \omega \end{pmatrix} V$$

$\Rightarrow$  Eigenfrequencies only increases by  $\sqrt{V}|_{(x,y)=0} = 1$

$\Rightarrow$  Eigenfrequencies are the same as in (c). 1 and  $\sqrt{3}$ .

J. (a)  $\begin{pmatrix} x \\ y \end{pmatrix}'' + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}' + \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\uparrow \hat{M}$        $\uparrow \hat{M}$

$$\vec{x} = \vec{A} e^{\alpha t} \Rightarrow (\alpha^2 + \alpha \hat{J} + \hat{M}) \vec{A} = 0$$

(b)  $\begin{vmatrix} \alpha^2 + 4\alpha + 3 & -2 \\ -2 & \alpha^2 + 4\alpha + 3 \end{vmatrix} = (\alpha^2 + 4\alpha + 3)^2 - 4 = 4 \Rightarrow \text{Solutions are } \frac{-2 \pm \sqrt{3}}{-2 \pm i}$

The real roots are negative  $\Rightarrow$  overdamped

The complex roots give underdamped motion.

(c)  $\begin{vmatrix} \alpha^2 + 4\alpha + 4 & -\epsilon \\ -\epsilon & \alpha^2 + 4\alpha + 4 \end{vmatrix} = (\alpha + 2)^2 - \epsilon^2 = 0 \Rightarrow (\alpha + 2)^2 = \pm \epsilon$

$$\alpha^2 + 4\alpha + 4 \mp \epsilon = 0 \Rightarrow \alpha = -2 \pm \sqrt{\pm \epsilon}$$

$$\alpha = -2 \pm \sqrt{\epsilon} \text{ and } \alpha = -2 \pm i\sqrt{\epsilon}$$

2 modes are overdamped. 2 modes underdamped  
with angular freq  $\sqrt{\epsilon}$ .

3. (a) Assume upper and lower particles displace by  $X_1$  and  $X_2$ , we have

$$mX_1'' = -kX_1 + mg - k(X_1 - X_2)$$

$$mX_2'' = -k(X_2 - X_1) + mg$$

$$\text{In equilibrium. } \begin{aligned} k(2X_1 - X_2) &= mg \\ k(X_2 - X_1) &= mg \end{aligned} \Rightarrow \begin{aligned} X_1 &= 2mg/k \\ X_2 &= 3mg/k \end{aligned}$$

$$\text{Define deviations from eq. as } \begin{aligned} x_1 &= X_1 - 2mg/k \\ x_2 &= X_2 - 3mg/k \end{aligned}$$

We get  $\boxed{\begin{aligned} m\ddot{x}_1 &= -kx_1 - k(x_1 - x_2) \\ m\ddot{x}_2 &= -k(x_2 - x_1) \end{aligned}}$  mg factors out.

$$\vec{x}'' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' = -\omega_0^2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{Let } \vec{x} = \vec{A} e^{i\alpha\omega_0 t}$$

$$\Rightarrow \alpha^2 \omega_0^2 \vec{A} = \omega_0^2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \vec{A} \Rightarrow \begin{pmatrix} -2 - \alpha^2 & 1 \\ 1 & -1 - \alpha^2 \end{pmatrix} = (\alpha^2 + 1)(\alpha^2 + 2) - 1 = 0$$

$$\Rightarrow \alpha^4 + 3\alpha^2 + 1 = 0. \quad \alpha^2 = \frac{1}{2}(-3 \pm \sqrt{5}), \quad \alpha = \pm \sqrt{\frac{1}{2}(3 \pm \sqrt{5})} i$$

$\boxed{\text{eigenfrequencies} = \sqrt{\frac{1}{2}(3 \pm \sqrt{5})} \omega_0 \equiv \omega_{\pm}}$

Solve eigenmodes:  $\begin{pmatrix} -2 - \frac{1}{2}(-3 \pm \sqrt{5}) & 1 \\ 1 & -1 - \frac{1}{2}(-3 \pm \sqrt{5}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \pm \frac{\sqrt{5}}{2} \end{pmatrix} \Rightarrow \vec{x}(t) = \begin{pmatrix} 1 \\ \frac{1}{2} \pm \frac{\sqrt{5}}{2} \end{pmatrix} (A \cos \omega_{\pm} t + B \sin \omega_{\pm} t) + \begin{pmatrix} 1 \\ \frac{1}{2} \mp \frac{\sqrt{5}}{2} \end{pmatrix} (C \cos \omega_{\mp} t + D \sin \omega_{\mp} t)$$

$$\Rightarrow \vec{x}'(t) = \begin{pmatrix} 1 \\ \frac{1}{2} \pm \frac{\sqrt{5}}{2} \end{pmatrix} (-A \omega_{\pm} \sin \omega_{\pm} t + B \omega_{\pm} \cos \omega_{\pm} t) +$$

$$\begin{pmatrix} 1 \\ \frac{1}{2} \mp \frac{\sqrt{5}}{2} \end{pmatrix} (-C \omega_{\mp} \sin \omega_{\mp} t + D \omega_{\mp} \cos \omega_{\mp} t)$$

$$\vec{x}(0) = 0 \Rightarrow A = C = 0.$$

$$x_1'(0) = 0 \Rightarrow B\omega_{\pm} + D\omega_{\mp} = 0$$

$$x_2'(0) = v_0 = \frac{1+\sqrt{5}}{2}B\omega_{\pm} + \frac{1-\sqrt{5}}{2}D\omega_{\mp} \quad \left. \begin{array}{l} v_0 = B\omega_{\pm} \left( \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \right) \\ \Rightarrow B = \frac{v_0}{\sqrt{5}\omega_{\pm}}, \quad D = -\frac{v_0}{\sqrt{5}\omega_{\mp}} \end{array} \right\}$$

$$\Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left( \frac{1}{1+\sqrt{\beta}} \right) \frac{v_0}{\sqrt{F}\omega} \sin \omega t - \left( \frac{1}{1-\sqrt{\beta}} \right) \frac{v_0}{\sqrt{F}\omega} \sin \omega t$$

Let  $m = m_1 = m_2$

$$m x_1'' = -k(x_1 - x_2)$$

$$m_2 x_2'' = k(x_1 - x_2) + k(x_3 - x_2)$$

$$m x_3'' = k(x_2 - x_3)$$

Define  $\omega = \sqrt{k/m}$  and assume  $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{A} e^{\alpha \omega t}$ ,  $\beta = m/m_2$

$$\Rightarrow \vec{x}'' = -\omega^2 \begin{pmatrix} 1 & -1 & 0 \\ -\beta & 2\beta & \beta \\ 0 & -1 & 1 \end{pmatrix} \vec{x} \Rightarrow \alpha^2 \vec{A} = - \begin{pmatrix} 1 & -1 & 0 \\ -\beta & 2\beta & \beta \\ 0 & -1 & 1 \end{pmatrix} \vec{A}$$

$$\Rightarrow \begin{vmatrix} 1+\alpha^2 & -1 & 0 \\ -\beta & \alpha^2+2\beta & \beta \\ 0 & -1 & 1+\alpha^2 \end{vmatrix} = 0 \Rightarrow (\alpha^2+1)(\alpha^2+2\beta) = 2\beta(\alpha^2+1) \\ \Rightarrow \alpha^2 = -1 \quad \text{or} \quad \alpha^4 + (2\beta+1)\alpha^2 = 0 \\ \Rightarrow \alpha = 0$$

$$\alpha = \pm \sqrt{1+2\beta} i$$

eigenfrequency eigenmode

$$1\text{st eigenmode } \alpha_1 = \pm i, \omega_1 = \omega = \sqrt{k/m} \Rightarrow \vec{A}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rightarrow \circlearrowleft \rightarrow \circlearrowright$$

$$2\text{nd eigenmode } \alpha_2 = 0 \quad \vec{A}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ translation} \rightarrow \circlearrowleft \rightarrow \circlearrowright \omega_2 = 0$$

$$3\text{rd eigenmode } \alpha_3 = \pm \sqrt{\alpha^2 + 1}. \omega_3 = \sqrt{\alpha^2 + 1} \sqrt{k/m}$$

$$\begin{pmatrix} \alpha^2+1 & -1 & 0 \\ -\beta & \alpha^2+2\beta & \beta \\ 0 & -1 & \alpha^2+1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -2\beta & -1 & 0 \\ -\beta & -1 & -\beta \\ 0 & -1 & -2\beta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \propto \begin{pmatrix} 1 \\ -2\beta \\ 1 \end{pmatrix} \rightarrow \circlearrowleft \rightarrow \circlearrowright$$