HW3 solution 2021P143b Cheng Chin

7. (a) First we determine a_0

$$\int_0^T f(t)\,dt = \int_0^T a_0\,dt = \frac{T}{2} \Rightarrow \boxed{a_0 = \frac{1}{2}}$$

2nd we determine a_n, $n = 1, 2, 3 \cdots$

$$\int_0^T \cos\frac{2\pi n t}{T} f(t)\,dt = \boxed{a_n \frac{T}{2} = 0}$$ since $f(t)$ is an odd func.

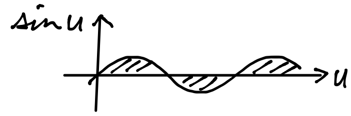
3rd we determine b_n

$$b_n = \frac{2}{T}\int_0^T \sin\frac{2\pi n t}{T} f(t)\,dt = \frac{2}{T}\frac{T}{2\pi n}\int_0^{\pi n} \sin u\,du$$

$= 2$ for $n = 1, 3, 5$

$= 0$ for $n = 2, 4 \cdots$

$$\Rightarrow \boxed{\begin{aligned} b_n &= \frac{2}{\pi n} \quad \text{for } n = 1, 3, 5 \cdots \\ &= 0 \quad \text{for } n = 2, 4, 6 \cdots \end{aligned}}$$



(b) $$c_n = \frac{1}{T}\int_0^T e^{-i2\pi n t/T} f(t)\,dt$$

$$= \frac{1}{T}\frac{T}{2\pi n}\int_0^{\pi n} e^{-iu}\,du = \frac{1}{2\pi n}\int (\cancelto{0}{\cos u} - i\sin u)\,du$$

$$= \frac{-i}{2\pi n}\int_0^{\pi n} \sin u\,du$$

$= \pm 2$ for $n = \pm 1, \pm 3, \pm 5 \cdots$

$= 0$ for $n = \pm 2, \pm 4 \cdots$

For $n = 0$, $c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}$

$$\Rightarrow \boxed{\begin{aligned} c_n &= 0 \quad \text{for } n = \pm 2, \pm 4 \cdots \\ &= -\frac{i}{\pi n} \quad \text{for } n = \pm 1, \pm 3, \pm 5 \cdots \\ &= \frac{1}{2} \quad \text{for } n = 0. \end{aligned}}$$

(c) $$f(\omega) = \frac{1}{2\pi}\int e^{-i\omega t} f(t)\,dt = \frac{1}{2\pi}\int e^{-i\omega t}\sum_n c_n e^{i2\pi n t/T}\,dt$$

$$= \sum_n \frac{c_n}{2\pi}\int e^{i(2\pi n/T - \omega)t}\,dt$$

$\int e^{i(k-k')x}\,dx = 2\pi\,\delta(k-k')$

$$\Rightarrow \boxed{f(\omega) = \sum_n c_n\,\delta(\omega - 2\pi n/T)}$$

(d) $c_n = \frac{1}{T}\int_0^T e^{-i2\pi n t/T} f(t)\,dt = \frac{1}{T}\int_0^T \cos 2\pi n t/T - i\sin 2\pi n t/T \, dt$

$= \frac{1}{2} a_n - i\frac{1}{2} b_n = \frac{1}{2}(a_n - i b_n)$

$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = a_0.$

$$\Rightarrow \boxed{\begin{array}{l} c_0 = a_0, \quad c_n = (a_n + i b_n)/2 \\ f(\omega) = \sum_n \frac{1}{2}(a_n + i b_n)\,\delta(\omega - 2\pi n/T) \end{array}}$$

2. (a). 1st. we find the period is π and the func. is even.

$\Rightarrow y(x) = \sum_{n=0} a_n \cos 2nx$

$a_0 = \langle y(x) \rangle = \int_0^\pi \sin x\,dx \,\frac{1}{\pi} = \frac{2}{\pi}$

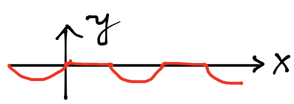
$a_n = \frac{2}{\pi}\int_0^\pi \cos 2nx \sin x\,dx = \boxed{\frac{4}{\pi}\,\frac{1}{1-4n^2}}$

(b) Function is odd

$\Rightarrow y(x) = \sum_{n=1} b_n \sin nx$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \; x\,dx = \frac{1}{\pi}(-2\pi \frac{1}{n} \cos n\pi) = \frac{-2}{n}(-1)^n$

$\Rightarrow \boxed{b_n = \frac{2}{n}(-1)^{n+1}}$

(c) 

$y = a_0 + \sum_{n=1} a_n \cos nx + b_n \sin nx.$

$a_0 = \langle y \rangle = -\int_0^\pi \sin x\,dx \,\frac{1}{2\pi} = -\frac{1}{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \; y(x) = \frac{-1}{\pi}\int_{\pi}^{2\pi} \cos nx \sin x\,dx$

$= 0 \quad$ for $n = 1, 3, 5$

$-\frac{2}{\pi}\,\frac{1}{n^2-1} \quad$ for $n = 2, 4, 6 \cdots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \; y(x) = -\frac{1}{\pi}\int_{\pi}^{2\pi} \sin nx \sin x\,dx$

$= -\frac{1}{\pi}\,\frac{\pi}{2}$ for $n = 1$, $\quad b_n = 0$ for $n = 2, 3 \cdots$

$$\boxed{y = -\frac{1}{\pi} - \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=\text{even}} \frac{1}{n^2-1}\cos nx}$$

(d) $x(t) = \int x(\omega) e^{i\omega t} d\omega$

$$\Rightarrow x'(t) = \int i\omega x(\omega) e^{i\omega t} d\omega$$

$$\Rightarrow x''(t) = \int -\omega^2 x(\omega) e^{i\omega t} d\omega$$

$$\Rightarrow \int (-\omega^2 + i\gamma\omega + \omega_0^2)\, x(\omega)\, e^{i\omega t} d\omega = \int f(\omega) e^{i\omega t} d\omega$$

Since Fourier transform is unique $\Rightarrow x(\omega) = \dfrac{f(\omega)}{\omega_0^2 - \omega^2 + i\gamma\omega}$

3. (a) Using inverse Fourier transform

$$f(\omega) = \frac{1}{2\pi}\int e^{-i\omega t} f(t)\, dt$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int e^{i\omega t} \frac{\int e^{-i\omega \tau} f(\tau) d\tau}{\omega_0^2 - \omega^2 + i\gamma\omega} d\omega = \frac{1}{2\pi}\iint \frac{e^{i\omega(t-\tau)} f(\tau)}{\omega_0^2 - \omega^2 + i\gamma\omega} d\omega\, d\tau$$

From 1(c), we have $f(\omega) = \sum_n c_n \delta(\omega - 2\pi n/T)$

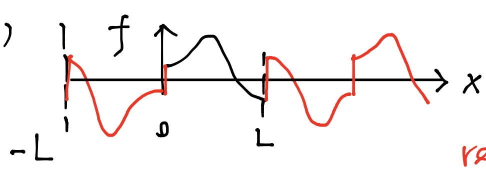
$$\Rightarrow x(t) = \sum_n c_n \int e^{i\omega t} \delta(\omega - 2\pi n/T) \frac{d\omega}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

$$= \sum_n c_n \frac{e^{i\omega_n t}}{\omega_0^2 - \omega_n^2 + i\gamma\omega_n} \qquad \omega_n \equiv 2\pi n/T$$

$$= \underbrace{\frac{1}{2\omega_0^2}}_{n=0} + \sum_{n=1,3,5\ldots} \frac{-i}{\pi n}\left[\underbrace{\frac{e^{i\omega_n t}}{\omega_0^2 - \omega_n^2 + i\gamma\omega_n}}_{n=1,3,5\ldots} - \underbrace{\frac{e^{-i\omega_n t}}{\omega_0^2 - \omega_n^2 - i\gamma\omega_n}}_{n=-1,-3,-5}\right]$$

$$\boxed{= \frac{1}{2\omega_0^2} + \sum_{n=1,3,5} \frac{2}{\pi n} \operatorname{Im} \frac{e^{i\omega_n t}}{\omega_0^2 - \omega_n^2 + i\gamma\omega_n}}$$

4. (a)



red line shows extended parts.

Since $f(x) = -f(-x)$ is odd, we only need all terms with sin. no cos.

Period is $2L$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{2\pi n x}{2L} \qquad b_n=\frac{2}{2L}\int_{-L}^{L}\sin\frac{2\pi n x}{2L} f(x)\,dx$$

$$=\frac{2}{L}\int_0^L \sin\frac{\pi n x}{L} f(x)\,dx$$

(b) $b_n=\frac{2}{L}\int_0^L \sin\frac{\pi n x}{L} f(x)\,dx$

$$=\frac{2}{L}\int_0^{L/4}\sin\frac{\pi n x}{L}\frac{4d}{L}x\,dx+$$

$$\frac{2}{L}\int_{L/4}^{L}\sin\frac{\pi n x}{L}\frac{4d}{3L}(L-x)\,dx$$

$$=\frac{8d}{L^2}\left(\int_0^{L/4} x\sin\frac{\pi n x}{L}dx+\frac{(-1)^{n+1}}{3}\int_0^{3L/4} x\sin\frac{\pi n x}{L}dx\right)$$

$$=\frac{8d}{L^2}\left(\left(\frac{L}{\pi n}\right)^2\sin\left(\frac{\pi n}{L}\frac{L}{4}\right)-\frac{L}{4}\frac{L}{\pi n}\cos\frac{\pi n}{4}+\right.$$

$$\left.(-1)^{n+1}\left(\frac{1}{3}\left(\frac{L}{\pi n}\right)^2\sin\frac{3\pi n}{4}-\frac{L}{4}\frac{L}{\pi n}\cos\frac{3\pi n}{4}\right)\right),$$ which reduces to

$$=\frac{8d}{3\pi^2 n^2}\left(3\sin\frac{\pi}{4}n+(-1)^{n+1}\sin\frac{3\pi n}{4}-\frac{3\pi n}{4}\cos\frac{\pi n}{4}-(-1)^{n+1}\frac{3\pi n}{4}\cos\frac{3\pi}{4}n\right)$$

$$=\frac{32d}{3\pi^2 n^2}\sin\frac{\pi n}{4}$$

$$\Rightarrow b_1=\frac{16\sqrt{2}}{3\pi^2}d \qquad b_2=\frac{8}{3\pi^2}d \qquad b_3=\frac{16\sqrt{2}d}{27\pi^2}$$

$$f(x)=\frac{8d}{3\pi^2}\left(2\sqrt{2}\sin\frac{\pi x}{L}+\sin\frac{2\pi x}{L}+\frac{2}{9}\sqrt{2}\sin\frac{3\pi x}{L}+\dots\right)$$

5. $\rho\partial_t^2\varphi+b\partial_t\varphi=E\partial_x^2\varphi$

(a) $\varphi\equiv Ae^{ikx}e^{i\omega t} \Rightarrow (-\omega^2\rho+bi\omega)\cancel{Ae^{ikx}e^{i\omega t}}$

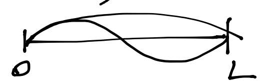
$$=-Ek^2\cancel{Ae^{ikx}e^{i\omega t}}$$

$$\Rightarrow \rho\omega^2-ib\omega-Ek^2=0. \quad \omega=\frac{1}{2\rho}\left(ib\pm\sqrt{4Ek^2\rho-b^2}\right)$$

Frequency is the real part of ω.

$$\mathrm{Re}[\omega] = \frac{1}{2\rho}\sqrt{4E\rho k^2 - b^2} = \sqrt{\frac{E}{\rho}k^2 - \frac{b^2}{4\rho^2}}$$

Since $\varphi(x=0) = \varphi(x=L) = 0.$



$$\Rightarrow \sin kL = 0. \quad k_n = n\pi/L. \quad n = 1, 2, 3 \cdots$$

$$\Rightarrow \omega_n = \sqrt{\frac{E}{\rho}k_n^2 - \frac{b^2}{4\rho^2}}$$

(b) general solution

$$\varphi(x,t) = \sum_{n=1}^{\infty} \varphi_n$$ ← wavefunction of n-th eigenmode.

$$\varphi_n = \sin k_n x \,(B_n e^{i\tilde{\omega}_n t} + C_n e^{-i\tilde{\omega}_n t})$$

$$= \sin k_n x \, e^{-bt/2\rho} (\underline{B_n} e^{i\omega_n t} + \underline{C_n} e^{-i\omega_n t})$$

each mode requires 2 constants given by initial position & velocity.

(c) $$\varphi(x,0) = \sum_{n=1} (B_n + C_n) \sin \frac{n\pi x}{2} = f(x)$$

$$\partial_t \varphi(x,0) = \sum_{n=1} i\omega_n (B_n - C_n) \sin \frac{n\pi x}{L} = g(x)$$

We can use Fourier sine series to write

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\frac{n\pi x}{L} \qquad B_n + C_n = f_n = \frac{2}{L}\int f(x) \sin\frac{n\pi x}{L}\,dx$$

$$g(x) = \sum_{n=1}^{\infty} g_n \sin\frac{n\pi x}{L} \quad \Rightarrow \quad B_n - C_n = -\frac{i}{\omega_n} g_n$$

$$\Rightarrow B_n = \frac{1}{2}\left(f_n - \frac{i}{\omega_n} g_n\right) = \frac{1}{L}\int_0^L \left[f(x) - \frac{i}{\omega_n} g(x)\right] \sin\frac{n\pi x}{L}\,dx$$

$$C_n = \frac{1}{2}\left(f_n + \frac{i}{\omega_n} g_n\right) = \frac{1}{L}\int_0^L \left[f(x) + \frac{i}{\omega_n} g(x)\right] \sin\frac{n\pi x}{L}\,dx$$

$$= B_n^*$$

$$\Rightarrow \varphi(x,t) = \sum_n \sin k_n x \,(B_n e^{i\omega_n t} + B_n^* e^{-i\omega_n t})$$

$$= \sum_n 2 \sin k_n x \,\mathrm{Re}[B_n e^{i\omega_n t}] \in \mathbb{R}.$$

(d) $\omega_n = K_n \left(E/\rho - b^2/4\rho^2 K_n^2\right)^{1/2}$ $\quad n = 1, 2, 3 \ldots$

$= K_n v_0 \left(1 - b^2/4\rho^2 K_n^2 v_0^2\right)^{1/2}$ $\qquad v_0 \equiv \sqrt{E/\rho}$

$= K_n v_0 \left(1 - b^2/8\rho^2 K_n^2 v_0^2\right)$ $\qquad K_n = n\pi/L = n K_1$

$= K_n v_0 - b^2/8\rho^2 K_n v_0$

Fundamental $\quad \omega_0 = K_1 v_0 - \underbrace{b^2/8\rho^2 K_1 v_0}_{\equiv \delta}$

1st overtone $\quad \omega_1 = 2 K_1 v_0 - \delta/2$

2nd overtone $\quad \omega_2 = 3 K_1 v_0 - \delta/3$

n.th overtone $\quad \omega_n = (n+1)\,\omega_1 + \delta\,(n+1 - 1/n+1)$

$= 0.$ for $n = 0$

$= 3/2$ for $n = 1$

$= 8/3$ for $n = 2$

$\to n$ for $n \gg 1$

$$\delta = \frac{b^2}{8\rho^2 K_1 v_0} \approx \frac{b^2}{8\rho^2}\frac{1}{\omega_0}$$

$$\delta_n = \frac{b^2}{8\rho^2 \omega_0}\,\frac{n(n+2)}{n+1}$$