2021 P143b HW4 Solution   Cheng Chin

1. (a) Assume $x = \epsilon > 0$

1. $\delta(\epsilon) = \lim_{\Delta\to 0} \frac{1}{\Delta} f(\frac{\epsilon}{\Delta}) = \lim_{\Delta\to 0} \frac{1}{\Delta}\frac{1}{\sqrt{\pi}} e^{-(\epsilon/\Delta)^2} = \lim_{x\to\infty} \frac{x}{\sqrt{\pi}\, e^{\epsilon^2 x^2}}$

Use L'Hospital's rule $= \lim_{x\to\infty} \frac{1}{2\sqrt{\pi}\, x\, e^{\epsilon^2 x^2}} = 0$

2. $\delta(0) = \lim_{\Delta\to 0} \frac{1}{\Delta}\frac{1}{\sqrt{\pi}} = \infty$

3. $\int \delta(x) dx = \lim_{\Delta\to 0} \int \frac{1}{\Delta}\frac{1}{\sqrt{\pi}} e^{-(x/\Delta)^2} dx = \lim_{\Delta\to 0} 1 = 1$

4. $\int f(u)\,\delta(x-u)\,du = \lim_{\Delta\to 0} \frac{1}{\Delta}\int f(u) \frac{1}{\sqrt{\pi}} e^{-(x-u)^2/\Delta^2} du$ $\quad y \equiv \frac{u-x}{\Delta}$, $u = x + \Delta y$

$= \lim_{\Delta\to 0} \frac{1}{\sqrt{\pi}} \int f(x+\Delta y)\, e^{-y^2} dy = \frac{1}{\sqrt{\pi}}\int f(x) e^{-y^2} dy = f(x).$

(b) 1. $\int g(x)\,\delta(ax+b)\,dx = \frac{1}{a}\int g(\frac{1}{a}(u-b))\,\delta(u)\,du = \frac{1}{a} g(-b/a)$

$u \equiv ax+b,\ x = \frac{1}{a}(u-b)$

2. $\int g(x)\,\delta'(x)\,dx = \int g(x)\, d\,\delta(x) = g(x)\delta(x)\Big|_{-\infty}^{\infty} - \int \delta(x)\, g'(x)\,dx = -g'(0)$

2. (a) Integrate $x'' + \omega_0^2 x = f(t)$ over $t = t_0 - \epsilon$ to $t_0 + \epsilon$

$$\Rightarrow \int_{t_0-\epsilon}^{t_0+\epsilon} x'' + \omega_0^2 x\, dt = x'(t_0+\epsilon) - x'(t_0-\epsilon) + \omega_0^2 x(t_0)\, 2\epsilon$$

$$\xrightarrow[\epsilon\to 0]{} x'(t_0^+) - x'(t_0^-) = v_0 = \int_{t_0-\epsilon}^{t_0+\epsilon} f(t)\,dt \Rightarrow f = v_0\,\delta(t-t_0)$$

Given the general solution $x = A\sin\omega_0 t + B\cos\omega_0 t$, it can be recast as $A\sin\omega_0(t-t_0) + B\cos\omega_0(t-t_0)$. Boundary conditions give

$x(t_0) = 0 \Rightarrow B = 0.$
$x'(t_0) = v_0 \Rightarrow A\omega_0 = v_0$ $\quad \Rightarrow x(t \geq t_0) = \frac{v_0}{\omega_0}\sin\omega_0(t-t_0)$

before the impulse $x(t<t_0) = 0.$

(b). $F(t) = \int F(z)\,\delta(t-z)\,dz$ is the summation of many delta functions.

For each one $F(z)\delta(t-z)$ the solution is given by (a) as

$$x_z = \begin{cases} 0, & t < z \\ \frac{F(z)}{\omega_0}\sin\omega_0(t-z), & t > z \end{cases}$$

The solution is then the sum of all such x_z for $z \in (-\infty, \infty)$

$$X_z'' + \omega_0^2 X_z = F(z)\,\delta(t-z) \quad \Rightarrow \int_{-\infty}^{\infty} dz\,(X_z'' + \omega_0^2 X_z) = F(t)$$

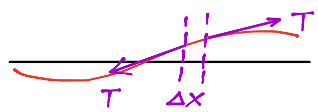
$$\Rightarrow X(t) = \int_{-\infty}^{\infty} X_z(t)\,dz = \underbrace{\int_{-\infty}^{t} X_z(t)\,dz}_{z<t.\ X_z \neq 0} + \underbrace{\int_{t}^{\infty} X_z(t)\,dz}_{z>t.\ X_z = 0}$$

$$= \int_{-\infty}^{t} \frac{F(z)}{\omega_0} \sin\omega_0(t-z)\,dz$$

Impulses occur at a later time $z>t$ cannot change the motion at t.

3. (a) Kinetic energy density $= \frac{1}{2} m v^2/\Delta x = \frac{1}{2}\rho(\partial_t \varphi)^2 \equiv \rho_K$

potential energy $\equiv U = \frac{1}{2} k \chi^2$, χ = extension of the ~~spring~~ string



$$= \sqrt{\Delta x^2 + \Delta y^2} - \Delta x$$
$$= \Delta x\left(\sqrt{1 + \Delta y^2/\Delta x^2} - 1\right)$$
$$\approx \frac{1}{2}\left(\Delta y/\Delta x\right)^2 \Delta x = \frac{1}{2}(\partial_x \varphi)^2 \Delta x$$

Restoring force = Tension $= \frac{dU}{dx} = k\chi = 2T$

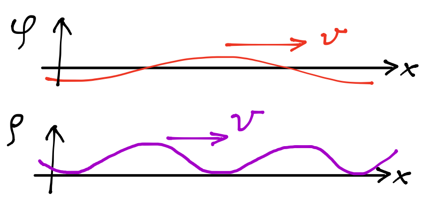
$\Rightarrow$ Potential energy density $= \frac{1}{2}(k\chi)\chi = \frac{1}{2} T (\partial_x \varphi)^2 \equiv \rho_U$

(b) $\varphi = A\cos k(x - vt)$

$$\rho_K = \frac{1}{2}\rho A^2 k^2 v^2 \sin^2 k(x-vt) = \frac{1}{2} T A^2 k^2 \sin^2 k(x-vt)$$

$$\rho_U = \frac{1}{2} T A k^2 \sin^2 k(x-vt) = \rho_K \Rightarrow \rho = \rho_K + \rho_U = T A k^2 \sin^2 k(x-vt)$$

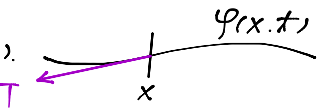
$$= 0 \quad \text{when } k(x-vt) = 0, \pi, 2\pi \cdots$$



Energy density also forms waves that propagates at speed v.

When we look at any spacetime $(x.t)$, energy density oscillates at freq $2kv$ when the energy drops to zero. the energy just moves to it's neighbors on the right hand side with velocity $v = \sqrt{T/\rho}$.

(c).



Consider position x. energy flow to the right must come from it's immediate neighbor on the left due to the tension T.

$$\Rightarrow j_E = dW/dt = \vec{T}\cdot d\vec{\varphi}/dt = T_y \partial_t \varphi \quad \text{(displacement only the } y\text{-dir.)}$$

$$= T_x \frac{T_y}{T_x}\partial_t \vec{\varphi} \approx -T\partial_x\varphi\,\partial_t\varphi \quad (\text{when } \partial_x\varphi \text{ is positive, } T_y<0)$$

$$(d)\ j_E = -T(-AK\sin K(x\pm vt))(\mp AKv\sin K(x\pm vt))$$

$$= \mp TA^2K^2v\sin^2 K(x\pm vt) = \mp \rho v$$

4. (a) $\varphi(0^-)=\varphi(0^+)$ $\quad A+B=C$ $\quad \Rightarrow \quad A+B=C$

$\varphi'(0^-)=\varphi'(0^+)$ $\quad Aik-Bik=Cik^*$ $\quad A-B=C(k^*/k)$

$$\Rightarrow C=\frac{2k}{k+k^*}A,\quad B=\frac{k-k^*}{k+k^*}A$$

(b) $\omega = kv_R = k^*v_L \Rightarrow k^*/k = v_R/v_L \Rightarrow C=\frac{2v_L}{v_L+v_R}A$ (transmission)

$$B=\frac{v_L-v_R}{v_L+v_R}A \quad \text{(reflection)}$$

At interface $x=0$. $j_A = TA^2k^2v_R\sin^2 kv_R t = TA^2k^2v_R\sin^2\omega t$

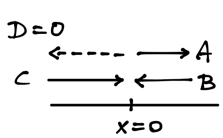
$$j_B = TB^2k^2v_R\sin^2 kv_R t = TB^2k^2v_R\sin^2\omega t$$

$$j_C = TC^2k^{*2}v_L\sin^2 k^*v_L t = TC^2k^2\frac{v_R^2}{v_L}\sin^2\omega t$$

So we need to prove $A^2 = B^2 + C^2\frac{v_R}{v_L} = A^2\left(\frac{v_L-v_R}{v_L+v_R}\right)^2 + A^2\left(\frac{2v_L}{v_L+v_R}\right)^2\frac{v_R}{v_L}$

$$= A^2\,\frac{(v_L-v_R)^2+4v_Lv_R}{(v_L+v_R)^2} = A^2$$

(c)



B and C are incident waves. A and D are outgoing waves. D=0 is the result of reflection from C and transmission from B.

Reflection from C has an amplitude of $\frac{v_R-v_L}{v_R+v_L}C = \frac{v_R-v_L}{v_R+v_L}\frac{2v_L}{v_L+v_R}A$

Transmission from B has an ampl. of $\frac{2v_L}{v_L+v_R}B = \frac{2v_L}{v_L+v_R}\frac{v_L-v_R}{v_L+v_R}A$

The two amplitudes cancel $\Rightarrow D=0$.

5. $\varphi = A\cos k(x-vt)$

(a) $P = P_0 - \frac{1}{\beta}\partial_x\varphi = P_0 + \frac{A}{\beta}k\sin k(x-vt)$ $\quad$ $0dB \rightarrow \Delta P = 20\mu Pa = Ak/\beta$

$\text{Intensity} = \text{energy flux/area} = TA^2k^2v\sin^2 k(x-vt)$ $\longleftarrow$ from 3(c)

$= \frac{1}{\beta}\Delta P^2\beta^2 v\sin^2 k(x-vt)$

$= \Delta P^2\beta v\sin^2 k(x-vt)$

string $\begin{matrix} T\rightarrow\beta \\ \rho\rightarrow n\end{matrix}$ sound

$\sqrt{T/\rho} \rightarrow \sqrt{1/n\beta}$

$= 331\,m/s$

$\langle\text{Intensity}\rangle = \frac{1}{2}\beta v\Delta P^2$

$= \frac{1}{2}\,7200/GPa\ 331\,m/s\,(20\times10^{-6}\,Pa)^2$

$= 4.7\times10^{-13}\,W/m^2$

(b) $P = P_0 + Ak/\beta\sin k(x-vt)$ & pressure cannot be negative

$P_0 = 1atm = 10^5\,Pa$ $\quad$ $0\,dB \Rightarrow 20\,\mu Pa$

$x\,dB \Rightarrow 20\mu * 10^{x/20} < 10^5\,Pa$

$\Rightarrow x < 194$

Pressure goes to zero above 194 dB.

(c) $A = \beta\Delta P/k = \beta\Delta P v/\omega$ $\quad$ $\omega = kv$

$(0dB) = 7200/10^9\ 20\times10^{-6}\ 331/2\pi\times100 = 7.5\times10^{-11}\,m$

$(100dB) = 10^5\times7.5\times10^{-11} = 7.5\,\mu m$ $\quad$ $\langle I\rangle = 4.7\times10^{-3}\,W/m^2$

$(200dB) = 10^{10}\times7.5\times10^{-11} = 75\,cm$ $\quad$ $= 4.7\times10^{7}\,W/m^2$