

# HW5 Solution Cheng Chin

1 a)  $A_1 = (e^y, 1, e^x)$

$$\nabla \cdot A_1 = \partial_x e^y + \partial_y 1 + \partial_z e^x = 0 \quad \text{no source/sink}$$

$$\nabla \times A_1 = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ \partial_x & \partial_y & \partial_z \\ e^y & 1 & e^x \end{vmatrix} = \partial_x \cdot 0 - \partial_y \partial_z - \partial_z \partial_y = (0, -e^x, -e^y) \quad \text{circulation } \checkmark$$

$$A_2 = (z^2 x, 0, -z x^2) \quad \nabla \cdot A_2 = z^2 - x^2 \quad \text{source/sink } \checkmark$$

$$\nabla \times A_2 = (0, 2xz + 2xz, 0) = 4xz(0, 1, 0) \quad \text{circulation } \checkmark$$

$$A_3 = (\cos y, \sin z - \sin x, -\cos y) \quad \nabla \cdot A_3 = 0 \quad \text{no source/sink}$$

$$\nabla \times A_3 = (\sin y, 0, -\cos x - \sin y) \quad \text{circulation } \checkmark$$

b.  $\nabla \cdot (\phi \vec{A}) = \partial_i(\phi A_i) = A_i \partial_i \phi + \phi \partial_i A_i$   
 $= \phi \nabla \cdot \vec{A} + (\vec{A} \cdot \vec{\nabla}) \phi$

$$\begin{aligned} \nabla \times (\nabla \times \vec{A}) &= \epsilon_{ijk} \partial_i \partial_j \epsilon_{klm} \partial_l A_m = \epsilon_{kij} \epsilon_{klm} \partial_i \partial_j \partial_l A_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_i \partial_j \partial_l A_m = \partial_i \partial_j \partial_l A_j - \partial_i \partial_j \partial_l A_i \\ &= \vec{\nabla}(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \end{aligned}$$

$$\begin{aligned} \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) &= \partial_i(\phi \partial_i \psi - \psi \partial_i \phi) \\ &= \partial_i \phi \partial_i \psi + \phi \partial_i^2 \psi - \partial_i \psi \partial_i \phi - \psi \partial_i^2 \phi \\ &= \phi \nabla^2 \psi - \psi \nabla^2 \phi \end{aligned}$$

d. a)  $\psi = \omega D(K, x - \omega t) + \omega D(K_2 x - \omega_2 t)$   
 $= \omega D\left(\frac{K_1 + K_2}{2} x - \frac{\omega_1 + \omega_2}{2} t\right) + \omega D\left(\frac{K_1 - K_2}{2} x - \frac{\omega_1 - \omega_2}{2} t\right)$   
 this oscillates faster      this slower is the envelope function.  
 $v_p = \frac{\omega_1 + \omega_2}{K_1 + K_2}$        $v' = \frac{\omega_1 - \omega_2}{K_1 - K_2}$

b) Given  $\omega = \omega(k)$ ,  $v' = \frac{\omega(k_1) - \omega(k_2)}{k_1 - k_2} \xrightarrow{k_1 \approx k_2} \omega'(k) = v_g$ .

c)  $\omega(k) = v k (1 + \epsilon k) \Rightarrow v_p = \frac{\omega}{k} = v(1 + \epsilon k) = v(1 + \epsilon \frac{\omega}{v_p})$

$$v_L = v(1 + \epsilon \frac{\omega_L}{v_L}) \Rightarrow v_L(1 + \epsilon \frac{\omega_R}{v_R}) = v_R(1 + \epsilon \frac{\omega_L}{v_L})$$

$$v_R = v(1 + \epsilon \frac{\omega_R}{v_R})$$

$$\Rightarrow \epsilon = -\frac{v_R - v_L}{\frac{v_L}{v_R} \omega_R - \frac{v_R}{v_L} \omega_L} = 460 \times 10^{-6} \text{ m} = 4.6 \mu\text{m} \Rightarrow v = 343.259 \text{ m/s}$$

$$\frac{d\omega}{dk} = v(1 + \epsilon k) + v \leq k = v(1 + 2 \leq k) = 343.26 \text{ m/s} \quad \text{at } 47.5 \text{ Hz} \\ = 343.50 \text{ m/s} \quad \text{at } 4186 \text{ Hz.}$$

3. a)

$$\lambda = v_p / \omega_0$$

Assume the  $N$ -th wavefront reaches the car @  $t=0$ , the next wavefront reaches the car @  $t$ :  $v_p t = vt + \lambda$

$$\Rightarrow t = \frac{\lambda}{v_p - v} = \frac{1}{\omega_0} \frac{v_p}{v_p - v} \quad \text{At this time the } N\text{-th wavefront is } v_p t + vt \text{ away from the car}$$

$$\Rightarrow \text{reflected freq } \omega_r = \frac{v_p}{(v_p + v)t} = \omega_0 \frac{v_p - v}{v_p + v}.$$

b)

At  $t=0$ ,  $N$ th wavefront reaches the car. the next wavefront reaches the car @  $t$ :

$$(v_p + \omega)t = vt + (v_p + \omega)/\omega_0 \Rightarrow t = \frac{1}{\omega_0} \frac{v_p + \omega}{v_p + \omega - v}$$

At this time the  $N$ th wavefront is  $(v_p - \omega)t + vt$  away from the car.

$$\Rightarrow \text{ref. freq } \omega_r = \frac{v_p - \omega}{v_p - \omega + v} \frac{1}{t} = \omega_0 \frac{v_p - \omega}{v_p + \omega} \frac{v_p + \omega - v}{v_p - \omega + v}$$