

HW8 Solution Cheng Chin

$$1. (a) C_V = \frac{1}{M} \partial_T Q|_V = \frac{1}{M} \partial_T U = \frac{1}{M} \gamma N K = \gamma \frac{1}{m} K$$

$$\Rightarrow U = \gamma N K T = C_V N m T$$

$$(b) \beta = -\frac{1}{V} \partial_P V|_T = -\frac{1}{V} \partial_P \frac{NKT}{P}|_T = \frac{1}{V} \frac{NK T}{P^2} = \frac{1}{P}$$

$$\begin{aligned} \beta_s &= -\frac{1}{V} \partial_P V|_S \\ &= -\frac{1}{V} \frac{1}{\partial_P} \\ &= \frac{\gamma}{\gamma+1} P \end{aligned} \quad \begin{aligned} \text{isentropic process: } dU &= T \cancel{\partial S} - P dV \xrightarrow{0} dV = -\frac{1}{P} dU = -\frac{\gamma}{P} dP \\ &\Rightarrow P dV = -\gamma P dV - \gamma V dP \\ &\Rightarrow (\gamma+1) P dV = -\gamma V dP \\ &\Rightarrow \partial_P V|_S = -\frac{\gamma}{\gamma+1} \frac{V}{P} \end{aligned}$$

Sound vibrates at freq reaching 10kHz, which means the air is partly compressed and decompressed within $\ll 1\text{ms}$ time scale. Not enough time

for air molecules to reach new equilibrium. $\Rightarrow \beta = \beta_s$.

$$(c) \alpha_p = \frac{1}{V} \partial_V P|_p = \frac{1}{V} \partial_T NKT/P = \frac{NK}{PV} = \frac{1}{T}$$

$$\alpha_V = \frac{1}{P} \partial_T P|_V = \frac{1}{P} \partial_T \frac{NKT}{V} = \frac{NK}{PV} = \frac{1}{T}$$

$$(d) \text{mean distance } l = n^{-1/3} = \left(\frac{N}{V}\right)^{-1/3} = \left(\frac{P}{KT}\right)^{-1/3} = \left(\frac{KT}{P}\right)^{1/3}$$

$$dl = \frac{1}{3} \left(\frac{KT}{P}\right)^{1/3} \frac{\partial T}{T} \Rightarrow \frac{dl}{l} = \frac{1}{3} \frac{\partial T}{T} = \frac{1}{3} \frac{1}{300} = \frac{1}{900} \approx 0.11\%$$

$$2. (a) \Delta U = -P \Delta V + \cancel{\Delta Q} \xrightarrow{0} \Rightarrow (\gamma+1) P \Delta V = -\gamma V \Delta P$$

$$\Rightarrow \frac{\Delta V}{V} = -\frac{\gamma}{\gamma+1} \frac{\Delta P}{P}$$

$$\Rightarrow \ln P = -\frac{\gamma+1}{\gamma} \ln V + C' = \ln V^{-\frac{\gamma+1}{\gamma}}$$

$$\Rightarrow PV^{\frac{\gamma+1}{\gamma}} = C$$

Since $\Delta Q = T \Delta S = 0 \Rightarrow S = \text{const. } C$ and be related to S .

$$(b) \Delta S = \frac{dU}{T} + \frac{P}{T} dV \Rightarrow \Delta S = \int \frac{\gamma N K dT}{T} + \int \frac{N K}{V} dV \quad U = C_V N m T = \gamma N K T$$

$$= \gamma N K \ln \frac{T_2}{T_1} + N K \ln \frac{V_2}{V_1} = N K \ln \frac{T_2 V_2}{T_1 V_1}$$

$$= NK \ln \frac{P_2^{\gamma} V_2^{\gamma+1}}{P_1^{\gamma} V_1^{\gamma+1}} = C_v N m \ln \frac{P_2 V_2^{(\gamma+1)/\gamma}}{P_1 V_1^{(\gamma+1)/\gamma}}$$

$$(c). S_2 - S_1 = NK \ln \frac{V_2 V_2^{3/2}}{V_1 V_1^{3/2}} = NK \ln \frac{V_2 (P_2 V_2)^{3/2}}{V_1 (P_1 V_1)^{3/2}} = \frac{3}{2} NK \ln \frac{P_2 V_2^{3/2}}{P_1 V_1^{3/2}}$$

This is identical to (b) for $\gamma = 3/2$

$$\alpha = \frac{4\pi m}{3h^2}, \gamma = 3/2$$

$$(d). \text{ Given } S = NK \ln \frac{V}{N} \left(\frac{4\pi m V}{3h^2 N} \right)^\gamma = NK \ln \alpha \frac{V^\gamma P^\gamma V^{\gamma+1}}{N^{\gamma+1}} = \gamma NK \ln \alpha V P \left(\frac{V}{N} \right)^{\gamma+1}$$

$$C = PV^{(\gamma+1)/\gamma} = N^{\frac{\gamma+1}{\gamma}} \frac{1}{\alpha \gamma} e^{S/NK} = \frac{3h^2}{4\pi m} \frac{1}{\gamma} N^{\frac{\gamma+1}{\gamma}} e^{S/NK}$$

$$3. a) \int p(x) dx = A \int e^{-mx^2/2kT} dx = A \sqrt{\frac{2\pi kT}{m}} = 1 \Rightarrow A = \left(\frac{m}{2\pi kT} \right)^{1/2}$$

$$p(\vec{v}) = \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-m(v_x^2 + v_y^2 + v_z^2)/2kT}$$

$$b) P(v) dv = p(\vec{v}) 4\pi v^2 dv \Rightarrow p(v) = 4\pi v^2 \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-mv^2/2kT}$$

$$c) \langle v^2 \rangle = \int v^2 p(v) dv = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \int_0^\infty v^4 e^{-mv^2/2kT} dv = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \frac{3}{8} \frac{\sqrt{\pi}}{(m/2kT)^{5/2}}$$

$$= \frac{3}{2} \pi \frac{2kT}{m} = \frac{3kT}{m} \Rightarrow V_{rms} = \sqrt{\frac{3kT}{m}} \Rightarrow \frac{1}{2} m V_{rms}^2 = \frac{3}{2} kT$$

$$d) V_p = \frac{1}{\sqrt{m/\beta_s}} = \frac{1}{\sqrt{m}} \sqrt{\frac{\gamma+1}{\gamma}} \sqrt{P} = \sqrt{\frac{\gamma+1}{\gamma}} \sqrt{\frac{kT}{m}} = \sqrt{\frac{5/3}{m}} < \sqrt{\frac{3kT}{m}}$$

mass density

Sound speed is always less than rms thermal velocity.

$$\text{In fact } \int_0^{V_p} p(v) dv = \int_0^{V_p} 4\pi v^2 V_{rms}^{-3} \left(\frac{3}{2\pi} \right)^{3/2} e^{-\frac{3}{2} v^2 / V_{rms}^2} dv$$

$$= \int_0^{\epsilon} 4\pi \left(\frac{3}{2\pi} \right)^{3/2} u^2 e^{-\frac{3}{2} u^2} du \quad \begin{aligned} u &\equiv v/V_{rms} \\ \epsilon &\equiv V_p/V_{rms} = \sqrt{5/3} \end{aligned}$$

$$= 36\%.$$

\Rightarrow 64% of the air molecules are supersonic
at all temperatures.

4. a) In Carnot cycle energy extracted from T_H is $\propto T_H$, energy delivered to T_C is proportional to $T_C \Rightarrow$ efficiency $= \frac{\epsilon_H - \epsilon_C}{\epsilon_H} = \frac{T_H - T_C}{T_H} = 1 - \frac{T_C}{T_H}$ is the efficiency

Here if works reversely, electricity from the wall is $\propto T_H - T_c$, energy extracted from T_c is $\propto T_c \Rightarrow \frac{T_H - T_c}{T_c} = \frac{T_H}{T_c} - 1$, is the energy needed to remove unit energy from T_c .

Higher efficiency when $T_H \approx T_c$

$$b) \frac{T_H}{T_c} - 1 = \frac{295.6K}{277.6K} - 1 = 0.065J$$

$$c) dW = \left(\frac{T_H}{T} - 1\right) dQ = -\left(\frac{T_H}{T} - 1\right) C_v m k dT$$

$$W = \int_{T_c}^{T_c} -\left(\frac{T_H}{T} - 1\right) C_v m k dT = C_v m k \int_{T_c}^{T_H} \frac{T_H}{T} - 1 dT = C_v m k \left(T_H \ln \frac{T_H}{T_c} - 1 + T_c\right)$$

$$d) \lim_{T_c \rightarrow 0} W = C_v m k T_H \ln \frac{T_H}{T_c} \rightarrow \infty$$