

Midterm Solution 2021P143b Cheng Chin

$$\begin{aligned} m x_1'' &= -k_1 x_1 - k_2(x_1 - x_2) - \beta x_1' \\ m x_2'' &= -k_2(x_2 - x_1) - m_2 g x_2 / L \end{aligned}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' = \begin{pmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - m_2 g / L \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{\beta}{m} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'$$

$$(a) \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' = \begin{pmatrix} -\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -\omega_0^2 - g/L \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{Define } \gamma = g/\omega_0^2 L \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' = \omega_0^2 \begin{pmatrix} -2 & 1 \\ 1 & -1-\gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{Ansatz: } \vec{x} = \vec{A} e^{i\omega t} \Rightarrow -\omega^2 \vec{A} = \omega_0^2 \begin{pmatrix} -2 & 1 \\ 1 & -1-\gamma \end{pmatrix} \vec{A}$$

$$\Rightarrow \begin{vmatrix} -2 + \omega^2/\omega_0^2 & 1 \\ 1 & -1-\gamma + \omega^2/\omega_0^2 \end{vmatrix} = 0 \quad \text{let } x = \frac{\omega^2}{\omega_0^2}$$

$$\Rightarrow (x-2)(x-1-\gamma) = 0$$

$$\Rightarrow x^2 - (3+\gamma)x + 1 + 2\gamma = 0$$

$$\Rightarrow x = \frac{1}{2} (2+3 \pm \sqrt{(2+3)^2 - 4(1+2\gamma)})$$

$$= \frac{1}{2} (2+3 \pm \sqrt{(2+1)^2 + 4}) = 1, 3 \Rightarrow \gamma = 1 = g/\omega_0^2 L$$

$$\Rightarrow L = g/\omega_0^2$$

$$(b). \text{ Given } \gamma = 1 \Rightarrow \begin{pmatrix} -2 + \omega^2/\omega_0^2 & 1 \\ 1 & -2 + \omega^2/\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

$$\omega = \omega_0 \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = x_2 \quad \xrightarrow{\text{in phase oscillation}}$$

$$\omega = \sqrt{3}\omega_0 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = -x_2$$

$\xrightarrow{\text{out of phase motion}}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_0 t} + B \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{3}\omega_0 t}$$

$$(c). \quad \vec{x}'' = \omega_0^2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \vec{f} e^{i\omega_0 t} \quad \vec{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$$\text{Let } \vec{x} = \vec{A} e^{i\omega_0 t} \Rightarrow -\omega^2 \vec{A} = \omega_0^2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{A} + \vec{f}$$

$$\Rightarrow \begin{pmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{pmatrix} \vec{A} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$$\begin{cases} (2\omega_0^2 - \omega^2)Ax - \omega_0^2 Ay = f \\ -\omega_0^2 Ax + (2\omega_0^2 - \omega^2)Ay = 0 \end{cases} \Rightarrow (2\omega_0^2 - \omega^2)Ax - \frac{\omega_0^2 \omega_0^2}{2\omega_0^2 - \omega^2} Ax = f$$

$$\Rightarrow Ax = \frac{2\omega_0^2 - \omega^2}{(2\omega_0^2 - \omega^2)^2 - \omega_0^4} f$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{f}{(2\omega_0^2 - \omega^2)^2 - \omega_0^4} \begin{pmatrix} 2\omega_0^2 - \omega^2 \\ -\omega_0^2 \end{pmatrix} \sin \omega t$$

$\therefore (a) \psi = A e^{ik(x-vt)} + B e^{ik(x+vt)} + C e^{-ik(x-vt)} + D e^{-ik(x+vt)}$

To satisfy the boundary conditions  $\psi(0,t) = 0$ .

The spatial part must be  $\sin kx$

$$\Rightarrow \psi = \sin kx (A e^{ikvt} + B e^{-ikvt})$$

General solution includes all superposition.

$$\Rightarrow \psi = \int A(k) \sin kx (A e^{ikvt} + B e^{-ikvt}) dk.$$

(b). Given  $f(t) = f \cos \omega t = \psi(L, t)$ . Only one mode is selected.

$$\psi = A \sin \frac{\omega}{v} x \cos \omega t. \text{ B.C. gives } A \sin \frac{\omega L}{v} = f$$

$$\Rightarrow \psi = f \csc \frac{\omega}{v} L \sin \frac{\omega}{v} x \cos \omega t \Rightarrow c = f \csc \frac{\omega L}{v}$$

$$(c) \text{ Use } f(t) = \int f(\omega) e^{i\omega t} d\omega. \quad f(\omega) = \frac{1}{2\pi} \int f(t) e^{-i\omega t} dt$$

From (b), the solution to  $f(t) = f e^{i\omega t}$  is

$$\psi = f \csc \frac{\omega L}{v} \sin \frac{\omega x}{v} e^{i\omega t}$$

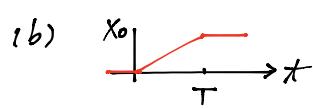
From superposition principle we have

$$\begin{aligned} \psi &= \int f(\omega) \csc \frac{\omega L}{v} \sin \frac{\omega x}{v} e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \iint f(z) \csc \frac{\omega L}{v} \sin \frac{\omega x}{v} e^{i\omega(z-t)} dz d\omega \end{aligned}$$

Take real part is the answer.

$$3. F = mx'' = -m\omega^2(x - x_0) \quad \text{introduce } u = x - x_0$$

$$(a) \Rightarrow u'' + x_0'' = -\omega^2 u \Rightarrow u'' = -\omega^2 u - x_0''$$



The force  $x''$  is non-zero only at  $t=0$  &  $t=T$   
The velocity experiences a jump of  $v_0$  &  $-v_0 \Rightarrow$

$$f(t) = v_0 \delta(t) - v_0 \delta(t-T), \quad v_0 = L/T$$

Solution to the 1st delta function is

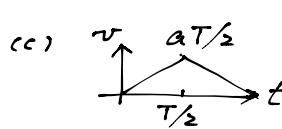
$$u = \frac{v_0}{\omega} \sin \omega t \quad \text{for } t > 0. \quad \text{s.t. } x(0) = v_0$$

Solution to the 2nd delta function is

$$u = -\frac{v_0}{\omega} \sin \omega(t-T), \quad t > T$$

$\Rightarrow$  Solution to  $f(t)$  with both delta functions is the sum of two.

$$x = x_0 + u = L + \frac{L}{\omega T} [\sin \omega t - \sin \omega(t-T)] = L + \frac{2L}{\omega T} \sin \frac{\omega T}{2} \cos \omega(t - \frac{T}{2})$$



Total travel distance =  $L = T/2 \cdot aT/2 = a(T/2)^2$

$$\Rightarrow \text{acceleration } a = 4L/T^2$$

$$\Rightarrow \text{deceleration} = -a$$

$$\Rightarrow x'' + \omega^2 x = f(t) = \begin{cases} a & 0 \leq t \leq T/2 \\ -a & T/2 \leq t \leq T \end{cases}$$

We may use result in HW 4, 2(b)

$$\begin{aligned} u(t) &= \int_{-\infty}^t \frac{f(z)}{\omega} \sin \omega(t-z) dz \\ &= \frac{a}{\omega} \int_0^{T/2} \sin \omega(t-z) dz - \frac{a}{\omega} \int_{T/2}^T \sin \omega(t-z) dz \\ &= \frac{a}{\omega^2} \left[ \omega \omega(t-z) \Big|_0^{T/2} - \frac{a}{\omega^2} \cos \omega(t-z) \Big|_{T/2}^T \right] \\ &= \frac{a}{\omega^2} \left[ \omega \omega(t-T/2) - \omega \omega t - \omega \omega(t-T) + \omega \omega(t-T/2) \right] \end{aligned}$$

$$\Rightarrow x = L + u = L + \frac{4L}{\omega^2 T^2} \left[ 2 \omega \omega(t-T/2) - \omega \omega t - \omega \omega(t-T) \right]$$

$$= L + \frac{8L}{\omega^2 T^2} \left[ \omega \omega(t-T/2) - \omega \omega(t-T/2) \cos \omega T/2 \right]$$

$$= L + \frac{16L}{\omega^2 T^2} \sin^2 \omega T/4 \cos \omega(t-T/2)$$