PHYS 143 – Problem Set 3

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a. We have $y(x) = |\sin x| = \begin{cases} \sin x & 2n\pi \leq x \leq (2n+1)\pi \\ -\sin x & (2n+1)\pi \leq x \leq (2n+2)\pi \end{cases}$

So the period is $L = \pi$. It is also an even function i.e. $y(x) = y(-x)$. This then implies that $b_n = 0$. So,

$$|\sin x| = a_0 + \sum_{n=1} a_n \cos\left(\frac{2n\pi x}{\pi}\right) \tag{1}$$

And so,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^\pi \sin(x)dx = \frac{2}{\pi} \\ a_n &= \frac{2}{\pi}\int_0^\pi \sin(x)\cos(2nx)dx = \frac{4}{\pi(1-4n^2)} \end{aligned} \tag{2}$$

Hence,

$$|\sin(x)| = \frac{2}{\pi} + \sum_{n=1} \frac{4}{\pi(1-4n^2)}\cos(2nx) \tag{3}$$

b. We are given $y(x) = \sum_n \delta(x-n)$ where $n \in \mathrm{Z}$ so the period is $L = 1$. We want to fourier expand this function i.e. write $y(x) = \sum_n \delta(x-n) = \sum_m c_m e^{2\pi i m x}$ where

$$\begin{aligned} c_m = \int_0^1 y(x)e^{-i2\pi m x}dx &= \sum_n \int_0^1 \delta(x-n)e^{-2\pi i m x}dx \\ &= \sum_n \int_{-n}^{1-n} \delta(x')e^{-2\pi i m x'}e^{2\pi i m n}dx' \\ &= \int_{-\infty}^{\infty} \delta(x')e^{-2\pi i m x'}dx = 1 \end{aligned} \tag{4}$$

where in the second line we made a change of variable to $x' = x - n$ so the limits of integration also changed accordingly. In the third line, we use the fact that $e^{2\pi i m n} = 1$ since m, n are both integers. In the third line, we use the fact that

$$\sum_{n=-\infty}^{n=\infty} \int_{-n}^{1-n} dx\, f(x) = \int_{-\infty}^{\infty} dx\, f(x) \tag{5}$$

So we get

$$y(x) = \sum_m e^{2\pi i m x} \tag{6}$$

c. We want to fourier transform $y(x) = \delta(x)$. We get

$$y(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = \frac{1}{2\pi} \tag{7}$$

d. $y(x) = \sum_n \delta(x - x_n)$. It's fourier transform will be

$$y(k) = \frac{1}{2\pi}\sum_n \int_{-\infty}^{\infty} \delta(x - x_n) e^{-ikx} dx = \frac{1}{2\pi}\sum_n e^{-ikx_n} \tag{8}$$

e. $f(x) = \int_{-\infty}^{\infty} dk\, f(k) e^{ikx}$. So,

$$\begin{aligned}\int dx\, f^*(x) f(x) &= \int dx \int dk'\, f^*(k') e^{-ik'x} \int dk\, f(k) e^{ikx} \\ &= \int dx \int dk' \int dk\, f^*(k') f(k) e^{-i(k'-k)x} \\ &= 2\pi \int dk' \int dk\, f^*(k) f(k) \delta(k - k') \\ &= 2\pi \int dk\, f^*(k) f(k)\end{aligned} \tag{9}$$

2.

a. We want to fourier expand the function $f(t) = \begin{cases} 1, & 0 \le t \le T/2 \\ 0, & T/2 \le t \le T \end{cases}$ We see that the function is odd i.e. $f(t) = -f(-t)$ so $a_n = 0$. And

$$a_0 = \frac{1}{T}\int_0^T f(t) dt = \frac{1}{T}\frac{T}{2} = \frac{1}{2} \tag{10}$$

And,

$$b_n = \frac{2}{T}\int_0^T f(t) \sin\frac{2\pi n t}{T} dt = \frac{2}{T}\int_0^{T/2} \sin\frac{2\pi n t}{T} dt = \begin{cases} \frac{2}{n\pi}, & (n = 1, 3, 5, \cdots) \\ 0, & (n = 0, 2, 4, \cdots) \end{cases} \tag{11}$$

b. We want to expand $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n t/T}$. Where

$$\begin{aligned} c_n &= \frac{1}{T}\int_0^T e^{-i2\pi n t/T} f(t) dt \\ &= \frac{1}{T}\int_0^{T/2} e^{-i2\pi n t/T} dt \\ &= \frac{1}{T}\frac{T}{2\pi n}\int_0^{n\pi} e^{-ix} dx \\ &= \frac{1}{2\pi n}\int_0^{n\pi} (\cos(x) - i\sin(x))\, dx = \frac{-i}{2\pi n}\begin{cases} \pm 2, & n = \pm 1, \pm 3, \cdots \\ 0, & n = \pm 2, \pm 4, \cdots \end{cases} \end{aligned} \tag{12}$$

where we made a change of variable in the third line $x = \frac{2n\pi t}{T}$. Also, for $n = 0$, $c_0 = \frac{1}{2}$ from the first line.

c. In this part, we want to fourier transform the function,

$$\begin{aligned} f(t) &= \int \tilde{f}(\omega) e^{i\omega t} d\omega \\ \implies \tilde{f}(\omega) &= \frac{1}{2\pi} \int dt f(t) e^{-i\omega t} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{n=\infty} c_n \int e^{i2\pi nt/T} e^{-i\omega t}\, dt \\ &= \frac{1}{2\pi} \sum_n c_n 2\pi \delta(\omega - \frac{2\pi n}{T}) \\ &= \sum_n c_n \delta(\omega - \frac{2\pi n}{T}) \end{aligned} \tag{13}$$

d. In this part, we want to express c_n and $\tilde{f}(\omega)$ as a function of a_n and b_n. We start with writing $f(t)$ in terms of cos and sin and $e^{\pm ix}$

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{i2\pi nt/T} = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \tag{14}$$

Now by expanding $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ and $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$ and comparing with the left hand side, we get

$$\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{1}{2}\left(a_n - ib_n\right) \end{aligned} \tag{15}$$

Since $f(\omega) = \sum_n c_n \delta\left(\omega - \frac{2\pi n}{T}\right) = \frac{1}{2} \sum \left(a_n - ib_n\right) \delta(\omega - \frac{2\pi n}{T})$.

3.

a. We want to solve the equation

$$x''(t) + \gamma x'(t) + \omega_0^2 x(t) = f(t) \tag{16}$$

Let us work in the Fourier transformed x

$$\begin{aligned} x(t) &= \int \tilde{x}(\omega) e^{i\omega t} d\omega \\ x'(t) &= \int i\omega \tilde{x}(\omega) e^{i\omega t} d\omega \\ x''(t) &= \int -\omega^2 \tilde{x}(\omega) e^{i\omega t} d\omega \end{aligned} \tag{17}$$

So the equation becomes

$$\int \left(-\omega^2 + i\gamma\omega + \omega_0^2\right) \tilde{x}(\omega) e^{i\omega t} d\omega = \int \tilde{f}(\omega) e^{i\omega t} d\omega$$
$$\implies \tilde{x}(\omega) = \frac{\tilde{f}(\omega)}{\omega_0^2 - \omega^2 + i\gamma\omega} \tag{18}$$

And so

$$\begin{aligned} x(t) = \int \tilde{x}(\omega) e^{i\omega t} d\omega &= \int \frac{\tilde{f}(\omega)}{\omega_0^2 - \omega^2 + i\gamma\omega} e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int d\omega \frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + i\gamma\omega} \int e^{-i\omega\tau} f(\tau) d\tau \\ &= \frac{1}{2\pi} \iint \frac{f(\tau) e^{i\omega(t-\tau)}}{\omega_0^2 - \omega^2 + i\gamma\omega} d\omega d\tau \end{aligned} \tag{19}$$

where we Fourier transformed $\tilde{f}(\omega)$.

b. In this part, we want to work with $f(t)$ from the previous question. We found $\tilde{f}(\omega) = \sum_n c_n \delta(\omega - \frac{2n\pi}{T})$. So,

$$\begin{aligned} x(t) &= \sum_n c_n \int d\omega \frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + i\gamma\omega} \delta\left(\omega - \frac{2n\pi}{T}\right) \\ &= \frac{1}{2\omega_0^2} + \sum_{n=1,3,5,\cdots} \frac{-i}{\pi n} \left[\frac{e^{i\omega_n t}}{\omega_0^2 - \omega_n^2 + i\gamma\omega_n} - \frac{e^{-i\omega_n t}}{\omega_0^2 - \omega_n^2 - i\gamma\omega_n}\right] \\ &= \frac{1}{2\omega_0^2} + \sum_{n=1,3,5,\cdots} \frac{2}{\pi n} \operatorname{Im}\left[\frac{e^{i\omega_n t}}{\omega_0^2 - \omega_n^2 + i\gamma\omega_n}\right] \end{aligned} \tag{20}$$

where we used $c_0 = \frac{1}{2}$ and $c_n = \frac{-i}{\pi n}$ with $n = \pm1, \pm3, \pm5, \cdots$.

4.

a. Let us work with the ansatz $\psi(x,t) = Ae^{ikx}e^{i\tilde{\omega}t}$

$$\begin{aligned} -\rho\tilde{\omega}^2 + ib\tilde{\omega} &= -Ek^2 \\ \rho\tilde{\omega}^2 - ib\tilde{\omega} - Ek^2 &= 0 \\ \implies \tilde{\omega} &= \frac{ib \pm \sqrt{-b^2 + 4E\rho k^2}}{2\rho} \\ \omega = \operatorname{Re}[\tilde{\omega}] &= \sqrt{\frac{E}{\rho}k^2 - \frac{b^2}{4\rho^2}} \end{aligned} \tag{21}$$

We are given the initial condition $\psi(0,t) = \psi(L,t) = 0$. We see that if $\psi(x,t) = \sin(kx)e^{i\omega t}$, the $\psi(0,t) = 0$ condition is satisfied. The other boundary condition,

$$\begin{aligned} &\psi(L,t) = 0 \\ &\sin(kL) = 0 \implies kL = n\pi, \ (n = 0, 1, 2, \cdots) \\ &k_n = \frac{n\pi}{L} \end{aligned} \tag{22}$$

The solution with $n = 0 \implies k_n = 0$ corresponds to the trivial solution $\psi(x,t) = 0$ where the string is at rest. The vibration modes of the string are given by $k_n = n\pi/L$ for $n = 1, 2, 3, \cdots$, or equivalently by $k_n = (n+1)\pi/L$ for $n = 0, 1, 2, 3, \cdots$

b. General solution

$$\begin{aligned} \psi(x,t) &= \sum_n \psi_n(x,t) \\ &= \sum_n \sin k_n x \left(B_n e^{i\tilde{\omega}_n t} + C_n e^{-i\tilde{\omega}_n t} \right) \\ &= \sum_n \sin(k_n x) e^{-bt/2\rho} \left(B_n e^{i\omega_n t} + C_n e^{-i\omega_n t} \right) \end{aligned} \tag{23}$$

From the intial conditions we have

$$\begin{aligned} \psi(x,0) &= \sum_n (B_n + C_n) \sin\left(\frac{n\pi x}{L}\right) = f(x) \\ \partial_t \psi(x,0) &= \sum_n i\omega_n \left(B_n - C_n\right) \sin\left(\frac{n\pi x}{L}\right) = g(x) \end{aligned} \tag{24}$$

This gives us

$$\begin{aligned} B_n &= \frac{1}{L} \int_0^L dx \left(f(x) - \frac{i}{\omega_n} g(x) \right) \sin\left(\frac{n\pi x}{L}\right) \\ C_n &= \frac{1}{L} \int_0^L dx \left(f(x) + \frac{i}{\omega_n} g(x) \right) \sin\left(\frac{n\pi x}{L}\right) = B_n^* \end{aligned} \tag{25}$$

So

$$\begin{aligned} \psi(x,t) &= \sum_n \sin k_n x \left(B_n e^{i\tilde{\omega}_n t} + C_n e^{-i\tilde{\omega}_n t} \right) \\ &= 2 \sum_n \sin\left(k_n x\right) \mathrm{Re}\left[B_n e^{i\omega_n t} \right] \end{aligned} \tag{26}$$

c. We found

$$\omega_n = \sqrt{\frac{E}{\rho} k_n^2 - \frac{b^2}{4\rho^2}} = \sqrt{\frac{E}{\rho} \left(\frac{(n+1)\pi}{L} \right)^2 - \frac{b^2}{4\rho^2}} \quad (n = 0, 1, 2, \cdots) \tag{27}$$

If $b = 0$,

$$\omega_n = \frac{E}{\rho} \frac{(n+1)\pi}{L} = \omega_0 (n+1) \quad (n = 0, 1, 2, \cdots) \tag{28}$$

Now consider b small

$$\omega_n^b = \sqrt{\frac{E}{\rho} \left(\frac{(n+1)\pi}{L} \right)^2 - \frac{b^2}{4\rho^2}} \simeq \omega_0 (n+1) - \frac{b^2}{8\rho^2} \frac{1}{\omega_0 (n+1)} \tag{29}$$

From here we see that

$$\omega_0^b = \omega_0 - \frac{b^2}{8\rho^2 \omega_0} \tag{30}$$

In terms of ω_0^b,

$$\begin{aligned}
\omega_n^b &= \omega_0(n+1) - \frac{b^2}{8\rho^2}\frac{1}{\omega_0(n+1)} \\
&= \left(\omega_0 - \frac{b^2}{8\rho^2\omega_0}\right)(n+1) + \left((n+1) - \frac{1}{(n+1)}\right)\frac{b^2}{8\rho^2\omega_0} \\
&\equiv \omega_0^b(n+1) + \delta_n
\end{aligned} \tag{31}$$