Physics 143A: Honors Waves, Optics, and Thermo

Spring Quarter 2024

Problem Set #4

Due: 11:59 pm, Thursday, April 17. Please submit to Canvas.

1. **Dirac's Delta function** *δ(x)* (10 points each)
   We may define Dirac's Delta function in the following
   - $f(x)$ is any function that has an integrated area of $\int f(x)dx = 1$.
   - Dirac's delta function is defined as $\delta(x) = \lim_{\Delta \to 0} \frac{1}{\Delta} f\left(\frac{x}{\Delta}\right)$

   (a) A common choice of f by physicists is the Gaussian function $f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$. Apply the above definition and prove that $\delta(x)$ satisfies the following properties
      1. $\delta(x \neq 0) = 0$
      2. $\delta(x = 0)$ diverges
      3. $\int \delta(x)dx = 1$
      4. $f(x) = \int f(u)\delta(x-u)du.$

   (b) Calculate the following
      1. $\int g(x)\delta(ax+b)dx$
      2. $\int g(x)\frac{d\delta(x)}{dx}\,dx, \quad g(\pm\infty) = 0$

2. **Kicked oscillator and Green's function (10 points each)**
   Let's consider a simple harmonic oscillator described by $x'' + \omega_0^2 x = f(t)$. The oscillator is at rest in the beginning $x(t=-\infty) = x'(t=-\infty) = 0$. At $t = 0$, you "kick" the oscillator with a short impulse. Immediately after the kick we have $x(0^+) = 0$ and $x'(0^+) = v_0$.

   (a) Show that such impulse can be expressed as $f(t) = v_0\delta(t)$ and the solution is

$$x_G(t) = \begin{cases} 0, & t < 0 \\ \dfrac{v_0}{\omega_0}\sin\omega_0 t, & t \geq 0 \end{cases}$$

   The impulse response of the system $x_G(t)$ is called Green's function.

   (b) **Green's function theorem** Now we consider a general driving force $F(t)$. The result of Question 1(a) 4 shows that a force can be expressed as a summation of impulses at all times, namely,

$$F(t) = \int F(\tau)\delta(t-\tau)d\tau.$$

   From the superposition principle, the solution is the summation of the responses to individual kicks. Show that the general solution of the oscillator, initially at rest and then driven by an arbitrary force $F(t)$, is given by

$$x(t) = \int_{-\infty}^{t} \frac{F(\tau)}{v_0} x_G(t-\tau)d\tau = \int_{-\infty}^{t} F(\tau)\frac{1}{\omega_0}\sin\omega_0(t-\tau)\,d\tau$$

   Hint: Given a kick of $f(t) = F(\tau)\delta(t-\tau)$ the solution would be $\frac{F(\tau)}{v_0}x_G(t-\tau)$.

3. Energy and energy flow in transverse waves (5 points each)

Here we will investigate how waves transport energy in a medium (string, air, water…). Assume the wave (transverse or longitudinal) satisfies the following wave equation

$$\rho\partial_t^2\psi(x,t) = T\partial_x^2\psi(x,t),$$

where ρ is the linear density of the medium and T is the tension in the medium. A traveling wave propagating along the $+x$ direction is given by $\psi(x,t) = A\cos k(x - vt)$, where $v = \sqrt{T/\rho}$.

(a) Consider a small section between x and $x + \Delta x$, show that the energy densities are given by

kinetic energy density: $\rho_K = \frac{1}{2}\rho(\partial_t\psi)^2$

potential energy density: $\rho_U = \frac{1}{2}T(\partial_x\psi)^2$.

(Hint: Kinetic energy is given by $\frac{1}{2}mV^2$ of the section. As for potential energy, think about how potential energy $\frac{kx^2}{2}$ is derived by stretching a spring. How much does the tension T extend the length of the section?)

(b) Given the traveling wave solution $\psi(x,t)$, calculate the total energy densities ρ_K and ρ_U. Show that the energy is propagating. At some point the total energy of the section becomes zero $\rho_E = \rho_K + \rho_U$=0. Where does the energy go?

(c) Derive and determine the energy flux of the traveling wave

energy flux: $j_E(x,t) = -T\partial_x\psi\partial_t\psi$.

(Hint: The energy flux is the energy that flows to the section comes from the work done by its neighboring sections through the tension force from $dW = T_y \cdot d\psi$, and use $\partial_x\psi = \frac{T_y}{T_x} \approx \frac{T_y}{T}$.)

(d) Show that for a traveling wave given by $\psi(x,t) = A\cos k(x \pm vt)$, we have

$$j_E = \pm v\rho_E.$$

This is a general result of all waves that the energy flux = energy density by the energy propagation velocity.