Physics 143A: Honors Waves, Optics, and Heat

Spring Quarter 2024
Problem Set #5
Due: 11:59 pm, Wednesday, April 24. Please submit to Canvas.

1. **(Math) Vector calculus (15 points each)**
   a) A vector field $\vec{A}(x, y, z)$ carries a source if $\nabla \cdot \vec{A} > 0$ and a sink if $\nabla \cdot \vec{A} < 0$. It carries circulation if $\nabla \times \vec{A}$ is a non-zero vector. Determine if the following fields carry source/sink or circulations?
   $\vec{A}_1 = (y^2 x, 0, -\mathrm{yx}^2)$
   $\vec{A}_2 = (\cos z, \sin x - \sin y, -\cos z)$

   b) Prove the following vector identities (ϕ, ψ are scalar fields, $\vec{A}$ is a vector field.)
   $\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi$
   $\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$

2. **Transmission and reflection of waves (10 points)**

Consider a wave moving toward an interface at $x = 0$, described by the equation

$$\partial_t^2 \psi(x, t) = v(x)^2 \partial_x^2 \psi(x, t),$$

where the wave propagation velocity changes across the interface

$$v(x < 0) = v_L$$
$$v(x > 0) = v_R.$$

Now consider an incident wave coming from the right side toward the interface

$$\psi_{in}(x > 0, t) = A \cos k(x + v_R t).$$

We may solve the equation with the following ansatz

$$\psi(x > 0, t) = A e^{ik(x + v_R t)} + B e^{ik(-x + v_R t)}$$
$$\psi(x < 0, t) = C e^{ik^*(x + v_L t)},$$

where B and C are the reflection and transmission amplitudes. At the end of the day, the solution is the real part of the ansatz.

A. Determine B and C in terms of A, k and k^* using the boundary conditions.

B. Since the wave equation respects the time reversal symmetry, flipping the time arrow $t \to -t$, the following should also be a solution

$$\psi(x > 0, t) = A e^{ik(x - v_R t)} + B e^{ik(-x - v_R t)}$$
$$\psi(x < 0, t) = C e^{ik^*(x - v_L t)}$$

However, here we have two waves with amplitudes B and C propagating toward the interface from both sides, but only one wave is coming out? Is such solution unphysical? Show that if you treat the terms with B and C as 2 independent incident waves, transmission of the B wave exactly cancels the reflection of the C wave, and thus no wave is propagating on left side $x < 0$ toward $x = -\infty$.

3. **Decibel scale of the strength of sound (5 points each)**

Alexander Bell, the inventor of telephone, introduced the unit of bel, which became decibel in acoustics: Zero decibel (0 dB) is defined as strength of the sound wave that produces $\pm 20\mu Pa$ in atmosphere air ($\mu Pa = 10^{-6} Pa$), which is also the typical limit of human hearing. Decibel is calculated in log scale such that every +20 dB corresponds to 10 times higher pressure. For instance, 20 dB corresponds to $200\mu Pa$ and 40 dB corresponds to $2mPa$ and so on. Human eardrum hearing is damaged above 100dB, corresponding to 2 Pa.

A. How much is the maximum displacement $\psi(x)$ of air molecules in the presence of acoustic waves at human's limit of 0dB and 100dB at frequency $\frac{\omega}{2\pi} = 100\text{Hz}$?
B. Show that sound in the air in principle cannot be louder than 200dB.
C. What is the energy intensity (in the unit of Watt/m^2) of the sound wave that could damage your eardrum?
(Hint: use 100 dB.)
D. For the same energy intensity, would the molecules displace more in water than in air?

Material	Density (kg/m^3)	Compressibility (1/GPa)
Air	1.22	7200
Water	1000	0.5
Copper	8960	0.0073

(GPa = 10^9 Pa.)

4. Electromagnetic (EM) waves in conductors. (15 points each)

EM waves propagate differently in metal than in insulators and in vacuum since electrons in metal can move and contribute to the EM fields. We will investigate how EM waves propagate in an ohmic conductor with charge density ρ and current density $j = \sigma E$, where σ is the conductivity. Maxwell equations are

$$\nabla \cdot E = \frac{\rho}{\epsilon}$$
$$\nabla \cdot B = 0$$
$$\nabla \times E = -\partial_t B$$
$$\nabla \times \mathrm{B} = \mu\epsilon\, \partial_t E + \mu j$$

A. Show that the electric field and magnetic field satisfy the wave equation:

$$\mu\epsilon\partial_t^2 E + \mu\sigma\partial_t E = \nabla^2 E - \frac{1}{\epsilon}\nabla\rho$$
$$\mu\epsilon\partial_t^2 B + \mu\sigma\partial_t B = \nabla^2 B$$

B. Assume there is no free charge $\rho = 0$, show that the solution of an EM wave propagating in the z direction can be written as

$$E(z,t) = E_0 e^{-\frac{z}{z_0}} e^{ik(z - v_p t)},$$

where $z_0 \geq 0$ is called the skin depth and determines the how far the field can penetrate into the conductor, and v_p is the phase velocity of the wave. Show that they are given by

$$v_p = \frac{\omega}{k} = \sqrt{\frac{2}{\mu\epsilon(1 + \sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}})}}$$
$$z_0 = \frac{2}{v_p \mu\sigma}.$$

(Hint: You may consider the conductor occupies the entire space $z \geq 0$ and the light propagates in vacuum from $z = -\infty$ toward the conductor.)

(Hint: You may assume the ansatz $E = e^{i(\tilde{k}z - \omega t)}$, where $\tilde{k} = k + i/z_0$ combines k and z_0 which may simplify your calculation. Other ansatzes should work just as well.)