Vector calculus

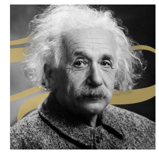
Consider a scalar A (like temperature) or a vector $\vec{A}$ (like electric field) defined in 3-dim space (x, y, z)

$$A(\vec{x}) = A(x, y, z)$$

$$\vec{A}(\vec{x}) = \hat{e}_1 A_1(\vec{x}) + \hat{e}_2 A_2(\vec{x}) + \hat{e}_3 A_3(\vec{x}) = \sum_{j=1,2,3} \hat{e}_j A_j \equiv \underline{\hat{e}_j}\, \underline{A_j} = \underline{\hat{e}_k}\, \underline{A_k}$$

Inner product: $\vec{A} \cdot \vec{B} = A_i B_i$

sum over repeated indexes



I said so

Gradient operator $\vec{\nabla} \equiv \hat{e}_i \partial_i = (\partial_x, \partial_y, \partial_z)$

$\vec{\nabla} A = \hat{e}_i \partial_i A = (\partial_x A, \partial_y A, \partial_z A)$ is a vector that indicates the shortest path to go up.

Divergence $\vec{\nabla} \cdot \vec{A} = (\partial_x, \partial_y, \partial_z) \cdot (A_x, A_y, A_z) = \partial_i A_i$ is a scalar that indicates if any source $(\vec{\nabla} \cdot \vec{A} > 0)$ or sink $(\vec{\nabla} \cdot \vec{A} < 0)$

Curl $\vec{\nabla} \times \vec{A} \equiv \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \hat{e}_x(\partial_y A_z - \partial_z A_y) + \hat{e}_y(\partial_z A_x - \partial_x A_z) + \hat{e}_z(\partial_x A_y - \partial_y A_x)$

$$\equiv \epsilon_{ijk} \hat{e}_i \partial_j A_k \quad \text{(sum over repeated } i, j, k)$$

Levi-Civita symbol

$\epsilon_{ijk} = 1$ if (ijk) is an even permutation of $(1,2,3)$ $\Rightarrow \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

$\epsilon_{ijk} = -1$ if (ijk) is an odd permutation of $(1,2,3)$ $\Rightarrow \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$

$\epsilon_{ijk} = 0$ if there is any repeated indexes $\Rightarrow \epsilon_{122} = \epsilon_{333} = \cdots = 0$

Example $\vec{A} \times \vec{B} = \epsilon_{ijk} \hat{e}_i A_j B_k = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} \hat{e}_i A_j B_k$

$\vec{\nabla} \times \vec{A} = \epsilon_{ijk} \hat{e}_i \partial_j A_k = \quad '' \quad \epsilon_{ijk} \hat{e}_i \partial_j A_k$

Now the most useful relation: $\epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$

Laplacian $\nabla^2 A = \vec{\nabla} \cdot \vec{\nabla} A = \hat{e}_i \partial_i (\vec{\nabla} A) = (\hat{e}_i \partial_i) \cdot (\hat{e}_j \partial_j A)$

$$= \delta_{ij} \partial_i \partial_j A = \partial_i \partial_i A = \partial_x^2 A + \partial_y^2 A + \partial_z^2 A$$

$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = 1$ when $i = j$; $= 0$ when $i \neq j$

Exercise:

Prove $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$ $\quad \hat{e}_l \partial_l \epsilon_{ijk} \hat{e}_i \partial_j A_k = \epsilon_{ijk} \underbrace{\hat{e}_l \cdot \hat{e}_i}_{\delta_{li}} \partial_l \partial_j A_k = \epsilon_{ijk} \underline{\partial_i \partial_j} A_k = 0$

$\vec{\nabla}\times\vec{\nabla}A = 0$ $\quad \epsilon_{ijk} e_i \partial_j (\vec{\nabla}A)_k = \epsilon_{ijk} e_i \partial_j \partial_k A_k = 0$

$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot\vec{\nabla}\times\vec{B}$ $\quad \hat{e}_i \partial_i (\epsilon_{jkl} \hat{e}_j A_k B_l)$

$$= \hat{e}_i \hat{e}_j \epsilon_{jkl} \partial_i (A_k B_l)$$
$$= \delta_{ij} \epsilon_{jkl} (A_k \partial_i B_l + B_l \partial_j A_k)$$
$$= \epsilon_{jkl} (A_k \partial_j B_l + B_l \partial_j A_k)$$
$$= A_k \epsilon_{jkl} \partial_j B_l + B_l \epsilon_{jkl} \partial_j A_k)$$
$$= - A_k \epsilon_{kjl} \partial_j B_l + B_l \epsilon_{ljk} \partial_j A_k$$
$$= \vec{B}\cdot\nabla\times\vec{A} - \vec{A}\cdot\nabla\times\vec{B}$$

$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})A + A(\nabla\cdot B) - (A\cdot\nabla)B - B(\nabla\cdot A)$

$$= \epsilon_{ijk} \hat{e}_i \partial_j (\vec{A}\times\vec{B})_k$$
$$= \epsilon_{ijk} \epsilon_{klm} \hat{e}_i \partial_j (A_l B_m)$$
$$= \epsilon_{kij} \epsilon_{klm} \hat{e}_i (A_l \partial_j B_m + B_m \partial_j A_l)$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$$
$$= \hat{e}_i (A_i \partial_j B_j + B_j \partial_j A_i - A_l \partial_l B_i - B_i \partial_j A_j)$$
$$= \vec{A}\,\vec{\nabla}\cdot\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}\cdot\vec{A})$$