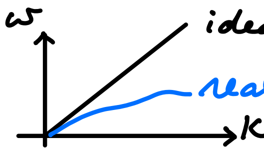
Dispersion. Group velocity. Doppler effect and shock wave

Dispersion: waves of diff freqs propagate differently.

In a medium, index of refraction $n = c/v$

$$\partial_t^2 \varphi = \frac{c^2}{n^2} \partial_x^2 \varphi \quad \text{is wavelength dependent.}$$

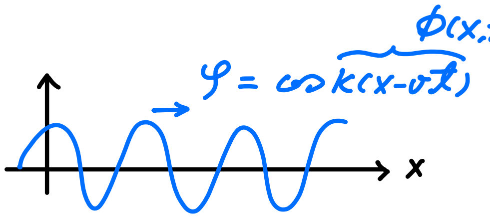
$$\Rightarrow \quad \omega = \frac{c}{n(\lambda)} k \equiv \omega(k)$$



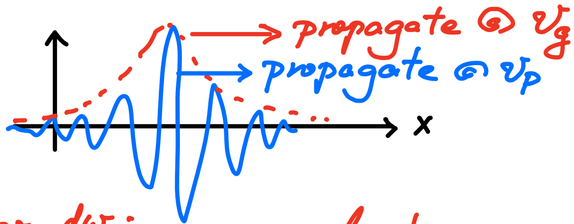
ideal: velocity is a const.

reality: velocity is not a const.

Dispersion: $\omega(k)$ or equivalently $v(\omega) \equiv \frac{c}{n}$ for EM waves.



$v_p = \frac{\omega}{k}$ is phase velocity.



$v_g = \frac{d\omega}{dk}$ is group velocity.

Let's analyze the general wave function centered at $k \approx k_0$. $\omega \approx \omega_0 = v_p k_0$

$$\varphi(x,t) = \int f(k,\omega) e^{i[kx - \omega(k)t]} dk$$

$$\approx \int f(k,\omega) e^{i[k_0 + (k-k_0)]x} e^{-i[\omega_0 + \omega'(k-k_0)]t} dk$$

$$= e^{i(k_0 x - \omega_0 t)} \int f(k,\omega) e^{i(k-k_0)x - i\omega'(k-k_0)t} dk$$

$$= e^{ik_0(x - v_p t)} \int f(k,\omega) e^{i(k-k_0)[x - \omega'(k)t]} dk$$

$$= e^{ik_0(x - v_p t)} F(x - \omega'(k)t)$$

$$= e^{ik_0(x - \underline{v_p} t)} F(x - \underline{v_g} t)$$

phase velocity group velocity

$$\varphi(x,0) = \int f(k,\omega) e^{ikx} dk$$

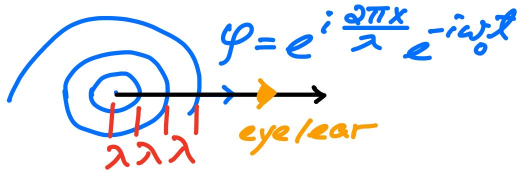
$$= e^{ikx} \underbrace{\int f(k,\omega) e^{i(k-k_0)x} dk}_{F(x) \text{ envelop}}$$

$$\omega(k) \approx \overset{\omega_0}{\omega(k_0)} + (k-k_0)\omega'(k_0) + \cdots$$

See text for other more informal arguments

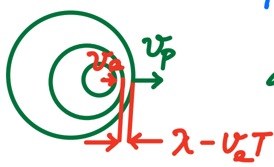
Doppler effect. Freq shifts due to

Stationary wave:



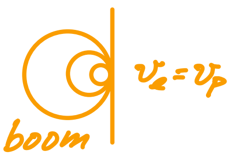
$$\text{apparent } \omega = \frac{2\pi}{T} = \frac{2\pi}{\lambda/v_p} = k v_p = \omega_0$$

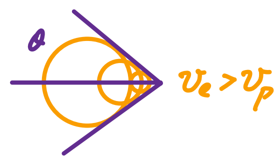
moving emitter:



$$\text{apparent } \omega = \frac{2\pi}{T} = \frac{2\pi}{(\lambda - v_e T)/v_p} = \frac{\omega_0}{1 - v_e T/\lambda} = \frac{\omega_0}{1 - v_e/v_p} > \omega_0$$

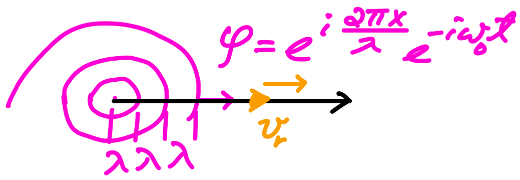
Shock waves: $v_e \geq v_p$





$$\sin\theta = \frac{v_p}{v_e}$$

moving receiver

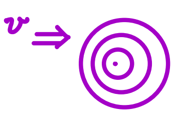


$$\text{apparent } \omega = \frac{2\pi}{\lambda/(v_p - v_r)} = \omega_0(1 - v_r/v_p) < \omega_0$$

both are moving

$$\omega = \frac{2\pi}{(\lambda - v_e T)/(v_p - v_r)} = \omega_0 \frac{1 - v_r/v}{1 - v_e/v}$$

moving medium



This is equivalent to both emitter and receiver moving with $-v$ in a stationary medium

$$\text{apparent } \omega = \omega_0 \frac{1 + v/v_p}{1 + v/v_p} = \omega_0$$

For light all of the above are incorrect in view of relativity.

Maxwell believed EM waves must be riding on some medium, called aether.
Aether needs to have many peculiar properties (low density, highly elastic...)

Michaelson-Morley exp suggests no such aether exists. $\Rightarrow$ Birth of special relativity.

principle of relativity: the result cannot depend on which one is considered at rest.

In the frame of source $\omega = \omega_0 (1 - v/c)$, but receiver's time is dilated by

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \Rightarrow \omega = \omega_0 (1 - v/c)\gamma = \omega_0 \sqrt{\frac{1 - v/c}{1 + v/c}}$$

In the frame of receiver $\omega = \omega_0 \frac{1}{1 + v/c}$, but source's time is dilated by γ

$$\Rightarrow \omega = (\omega_0/\gamma) \frac{1}{1 + v/c} = \omega_0 \sqrt{\frac{1 - v/c}{1 + v/c}}$$

relativistic formula $\omega = \omega_0 \sqrt{\frac{1 + \Delta v/c}{1 - \Delta v/c}}$ $\quad \Delta v = v_e - v_r$