Physics 143A: Honors Waves, Optics, and Heat

Spring Quarter 2024

Problem Set #1

Due: 11:59 pm, Wednesday, March 27. Please submit to Canvas.

1. **Math exercises** (5 points each)

   Ordinary Differential Equations

   (a) $x'' + 4x' + 3x = 0$ with initial condition $x(0) = 1$ and $v(0) = 1$.
   (b) $x'' + 2x' + 5x = 0$ with initial condition $x(0) = 0$ and $v(0) = -1$.
   (c) $x'' + 2x' + 5x = \sin t$ with initial condition $x(0) = -1/10$ and $v(0) = 0$.
   (d) $x' + 2x = \cos t$ with initial condition $x(0) = 1$.

   Complex numbers

   A complex number z can be expressed in the algebraic form $z = x + iy$ and the polar form $z = re^{i\theta}$, where $i^2 = -1$ and $x, y, r, \theta \in \mathbb{R}$ are the real part, imaginary part, modulus and argument, respectively. The two forms are linked by $x + iy = r(\cos\theta + i\sin\theta)$.

   (e) Simplify the following complex numbers
   $z_1 = \frac{i-4}{2i-3}$ in algebraic form,
   $z_2 = (1+i)^\alpha$ in polar form (α is a real number)
   $z_3 = \frac{1+i}{1-i} - (1+2i)(1+i)$ in algebraic form
   $z_4 = i^i$ in any form

   Approximations

   (f) Determine the asymptotic form of $f(x, y) = x^{3/2}(x + 4y)^{1/2} - (x + y)^2$ when $x \gg y$.
   (g) Simplify $f(x) = \frac{x - x_0}{\sqrt{\left(x^2 - x_0^2\right)^2 + 4\gamma^2 x^2}}$ when x is near x_0.
   (Hint: You may temporarily introduce $\epsilon \equiv x - x_0 \ll x, x_0$ and keep ϵ to leading order.)
   (h) A particle with mass $m = 1$ is trapped in the local minimum of the potential $V(x) = \frac{x-1}{x^2+3}$. Determine the position and potential energy of the minimum. Taylor expand the potential at the minima to the 2nd order (quadratic order).

2. **Damping an oscillator** (10 points each)
A high-quality factor *oscillator* is an oscillator that damps slowly and thus continues to ring even after you stop driving it. (Think about a high-quality sound fork or a music instrument.) One way to quickly damp out its motion is to connect it to a strong damper (like what the piano pedal does). Here we assume the oscillator follows the underdamped oscillator model $x'' + \gamma x' + \omega_0^2 x = 0$ with damping $\gamma \ll 2\omega_0$ and it oscillates with a non-zero amplitude of *A* at *t=0*.

At *t=0*, we suddenly apply a damper which increases the damping coefficient to $\gamma^* \gg \gamma$.

(a) A smart choice is to select the damper with a critical damping coefficient $\gamma^* \approx 2\omega_0$. Show that the time scale for total energy to decay by *e*=2.718… is given by $1/2\omega_0$.
(b) Why shouldn't an even larger damping, say, $\gamma^* \gg 2\,\omega_0$ yield faster damping? Let's consider the case $x(0) = 1$ and $x'(0) = 0$ and evaluate the potential energy $V = m\omega_0^2 x^2/2$ and kinetic energy $E_k = mx'^2/2$. Show that in the large damping limit we have

Kinetic energy $E_k(t) \propto e^{-\mu t}$

Potential energy $V(t) \propto e^{-\mu t}$,

and both energies decay slowly at the rate of $\mu \approx \frac{2\omega_0^2}{\gamma^*}$ for a large $\gamma^* \gg 2\,\omega_0$.
(c) Imagine we release a marble into a bowl filled with air, water or honey. Use your own words to explain why the energy of the marble decays slowly when the damping is both very small (in air) and very large (in honey)?

3. **Resonant energy transfer** (10 points each)
A high-Q oscillator can store an immense amount of energy. Let's consider the extreme case of a harmonic oscillator with negligible friction (thus $Q \to \infty$) and natural frequency ω_0.

(a) If we drive the oscillator with the external force $f\,cos\ \omega t$ at a frequency very close to the resonance $\omega = (1-\epsilon)\ \omega_0$ with $0 < \epsilon \ll 1$. Show that the total energy of the oscillator $E = E_k + V$ after the steady state is established is

$$E_{max} \approx \frac{mf^2}{8\omega_0^2\epsilon^2},$$

Note that approaching resonance, the energy of the oscillator diverges.

(b) If the oscillator is at rest initially $x(0) = x'(0) = 0$ and we start applying the driving force $cos(\omega t)$ at t=0, show that with $0 < \epsilon \ll 1$ the total energy averaged over 1 cycle increases from zero quadratically as

$$< E(t) >= \frac{1}{8} mf^2 t^2$$

(c) Combine the above results, show that it takes a long time to "charge up" a high-Q oscillator for small detuning $0 < \epsilon \ll 1$

$$T \sim \frac{1}{\omega\epsilon}$$

(Hint: Since $\epsilon \ll 1$, you can keep ϵ to leading order.)