PHYS 143 – Problem Set 1

Instructor: Cheng Chin

1

a. To solve the differential equation, assume $x = e^{\alpha t}$. Substituting this in the differential equation will give us

$$\begin{aligned} & x'' + 4x' + 3x = 0 \\ & \alpha^2 + 4\alpha + 3 = 0 \\ & (\alpha+1)(\alpha+3) \implies \alpha = -1,\, -3 \\ \implies & x(t) = Ae^{-3t} + Be^{-t} \end{aligned} \tag{1}$$

Now imposing the initial conditions gives us

$$\begin{aligned} & x(0) = A + B = 1 \\ & x'(0) = -3A - B = 1 \\ \implies & A = -1 \quad B = 2 \end{aligned} \tag{2}$$

b. Similarly for the second part, starting with the ansatz $x = e^{\alpha t}$ gives

$$\begin{aligned} & x'' + 2x' + 5x = 0 \\ & \alpha^2 + 2\alpha + 5 = 0 \\ & \alpha = -1 \pm \sqrt{-4} = -1 \pm 2i \\ & x(t) = Ae^{-t}e^{2it} + Be^{-t}e^{-2it} \end{aligned} \tag{3}$$

Imposing the initial conditions

$$\begin{aligned} & x(0) = A + B = 0 \\ & x'(0) = A(-1+2i) - B(1+2i) = -1 \\ & A = \frac{i}{4}\, B = \frac{-i}{4} \\ \implies & x(t) = \frac{-1}{2} e^{-t} \sin{(2t)} \end{aligned} \tag{4}$$

c. We now add a driving term to the previous equation. We already found the homogeneous solution in the previous part and now need to find the particular solution. For that, let us follow the same method from class. We note that $\sin(t) = \mathrm{Im}(e^{it})$. So we can solve for e^{it} and then take the imaginary part of that solution. Let $x_p = \mathrm{Im}(Ae^{it})$ so

$$\begin{aligned} & A(-1+2i+5)e^{it} = e^{it} \\ & A = \frac{2-i}{10} \\ \implies & x_p = \mathrm{Im}(x) = \frac{1}{5}\sin t - \frac{1}{10}\cos t \\ \implies & x = Ae^{-t}e^{2it} + Be^{-t}e^{-2it} + \frac{1}{5}\sin t - \frac{1}{10}\cos t \end{aligned} \tag{5}$$

We can impose the initial conditions on the full solution now

$$\begin{aligned} x(0) &= A + B - \frac{1}{10} = -\frac{1}{10} \implies A + B = 0 \\ x'(0) &= \frac{1}{5} + A(-1+2i) + B(-1-2i) = 0 \implies 2i(A-B) = -\frac{1}{5} \end{aligned} \tag{6}$$

So we finally get

$$x = \frac{1}{5}\sin t - \frac{1}{10}\cos t - \frac{1}{10}e^{-t}\sin 2t \tag{7}$$

d. Note that this is a first order differential equation. We will use the same steps as before. Let us first find the homogeneous solution assuming $x = e^{\alpha t}$

$$\begin{aligned} & x_h' + 2x_h = 0 \\ & x_h = Ae^{-2t} \end{aligned} \tag{8}$$

Now to find the particular solution, we note that $\cos(t) = \mathrm{Re} e^{it}$. So let $x_p = \mathrm{Re}(Be^{it})$

$$\begin{aligned} & B(i+2)e^{it} = e^{it} \implies B = \frac{2-i}{5} \\ & x_p = \mathrm{Re}(\frac{2-i}{5}e^{it}) = \frac{2}{5}\cos t + \frac{1}{5}\sin t \end{aligned} \tag{9}$$

Imposing the initial condition

$$x(0) = \frac{2}{5} + A = 1 \implies A = \frac{3}{5} \tag{10}$$

So that gives

$$x = \frac{2}{5}\cos t + \frac{1}{5}\sin t + \frac{3}{5}e^{-2t} \tag{11}$$

e. We can simplify the expressions as follows:

$$\begin{aligned}
z_1 &= \frac{i-4}{2i-3} = \frac{(i-4)(2i+3)}{(2i-3)(2i+3)} = \frac{14+5i}{13} \\
z_2 &= (1+i)^\alpha = \sqrt{2}^\alpha e^{i\alpha\pi/4} \\
z_3 &= \frac{1+i}{1-i} - (1+2i)(1+i) = \frac{(1+i)^2}{(1+i)(1-i)} - (1+2i)(1+i) = 1-2i \\
z_4 &= (e^{i\pi/2})^i = e^{-\pi/2}
\end{aligned} \tag{12}$$

f. Consider the function $f(x,y)$

$$\begin{aligned}
f(x,y) &= x^{3/2}(x+4y)^{1/2} - (x+y)^2 \\
&= x^2(1+4\frac{y}{x})^{1/2} - x^2(1+\frac{y}{x})^2 \\
&= x^2\left((1+4\frac{y}{x})^{1/2} - (1+\frac{y}{x})^2\right)
\end{aligned} \tag{13}$$

Since, $x >> y$, we have $t = y/x << 1$ and we can Taylor expand the function $\left((1+4t)^{1/2} - (1+t)^2\right)$

$$\begin{aligned}
&(1+4t)^{1/2} - (1+t)^2 \\
&= (1+2t-2t^2) - (1+2t+t^2) + \mathcal{O}(3) \\
&= -3t^2 + \mathcal{O}(t^3) \\
&= -3\left(\frac{y}{x}\right)^2 + \mathcal{O}((y/x)^3)
\end{aligned} \tag{14}$$

g. Let us write $x - x_0 \equiv \epsilon$. Then

$$\begin{aligned}
f(x) &= \frac{x-x_0}{\sqrt{(x^2-x_0^2)^2 + 4\gamma^2x^2}} \\
&= \frac{\epsilon}{\sqrt{\epsilon^2(2x_0+\epsilon)^2 + 4\gamma^2(x_0+\epsilon)^2}} \\
&= \frac{\epsilon}{2x_0\gamma} - \frac{\epsilon^2}{2x_0^2\gamma} + \mathcal{O}(\epsilon^3) \\
&= \frac{x-x_0}{2x_0\gamma} - \frac{(x-x_0)^2}{2x_0^2\gamma} + \mathcal{O}((x-x_0)^3)
\end{aligned} \tag{15}$$

h. Let us first determine the minima of the potential by finding the extrema of the potential $V'(x) = 0$

$$\begin{aligned}
V' &= \frac{1}{(x^2+3)} - \frac{(x-1)2x}{(x^2+3)^2} = 0 \\
&\implies x^2 + 3 - 2x(x-1) = 0 \\
&(x-3)(x+1) = 0 \\
&x = 3,\ x = -1
\end{aligned} \tag{16}$$

We then see that $V(3) = 1/6$ and $V(-1) = -1/2$. So the minima is at $x = -1$. Let us expand around this point.

$$\begin{aligned} V(x) &= V(-1) + V'(-1)(x+1) + \frac{1}{2}V''(-1)(x+1)^2 + \cdots \\ &= -\frac{1}{2} + \frac{1}{8}(x+1)^2 \end{aligned} \tag{17}$$

2.

a. Given that the oscillator is damped, the equation to work with would be

$$\begin{aligned} x'' + \gamma x' + \omega_0^2 x = 0 \\ x'' + \omega_0 x' + \omega_0^2 x = 0 \end{aligned} \tag{18}$$

Here in the second line, we used the fact that the oscillator is critically damped i.e. $\gamma \approx \omega_0$. We now know how to solve this equation- we will pick an ansatz $x = e^{\alpha t}$. So we get

$$\begin{aligned} &\alpha^2 + 2\omega_0\alpha + \omega_0^2 = 0 \\ &\alpha = -\omega_0 \\ \implies &x = Ae^{-\omega_0 t} \end{aligned} \tag{19}$$

To see how the energy decays, we first calculate the energy as a function of time using the solution above.

$$\begin{aligned} E(t) &= \frac{1}{2}mx'^2 + \frac{1}{2}m\omega_0^2 x^2 \\ &= \frac{1}{2}m\left(-A\omega_0 e^{-\omega_0 t}\right)^2 + \frac{1}{2}m\omega_0\left(Ae^{-\omega_0 t}\right)^2 \\ &= A^2 m\omega_0^2 e^{-2\omega_0 t} \end{aligned} \tag{20}$$

We want to find the time when $\frac{E(t)}{E(0)} = 1/e$

$$\begin{aligned} \frac{E(t)}{E(0)} &= \frac{1}{e} \\ e^{-2\omega_0 t} &= \frac{1}{e} \\ t &= \frac{1}{2\omega_0} \end{aligned} \tag{21}$$

Note that the other solution, $x = te^{-\omega_0 t}$ will also give the same result. This is because at late times, the fall off in time is exponential and that dominates.

b. In the case $\gamma >> 2\omega_0$, the equation will be the more general equation

$$\begin{aligned} &\alpha^2 + 2\omega_0\alpha + \omega_0^2 = 0 \\ &\alpha = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma - 4\omega_0^2}\right) = \frac{1}{2}\left(-\gamma \pm \gamma\sqrt{1 - \frac{4\omega_0^2}{\gamma}}\right) \end{aligned} \tag{22}$$

Given that $\gamma >> \omega_0$, we can taylor expand the solution above

$$\begin{aligned} &\alpha \approx \frac{1}{2}\gamma\left(-1 \pm (1 - 2\frac{\omega_0^2}{\gamma^2})\right) \\ \Longrightarrow\ &\alpha = -\frac{\omega_0^2}{\gamma},\ \alpha = -\gamma \\ \Longrightarrow\ &x(t) = Ae^{-(\omega_0^2/\gamma)t} + Be^{-\gamma t} \end{aligned} \tag{23}$$

We can now impose the initial conditions to fix A and B

$$\begin{aligned} &x(0) = A + B = 1 \\ &x'(0) = -\frac{\omega_0^2}{\gamma}A - \gamma B = 0 \\ \Longrightarrow\ &A = 1,\ B = 0 \end{aligned} \tag{24}$$

So we see that only the slowly decaying term is left $x = e^{-(\omega_0^2/\gamma)t}$. Given this, let us calculate the potential energy and kinetic energy

$$\begin{aligned} V &= \frac{1}{2}m\omega_0^2x^2 = \frac{1}{2}m\omega_0^2e^{-2(\omega_0^2/\gamma)t} \equiv V(0)e^{-2(\omega_0^2/\gamma)t} \\ K &= \frac{1}{2}mx'^2 = \frac{1}{2}m\omega_0^2e^{-2(\omega_0^2/\gamma)t} \equiv K(0)e^{-2(\omega_0^2/\gamma)t} \end{aligned} \tag{25}$$

So we see that both energies decay exponentially with time with a rate $\mu = \frac{2\omega_0^2}{\gamma}$.

c. In air, the damping is very low and honey is the other extreme with very large damping. So the damping force on the marble in honey will be large and it will barely move. On the other hand, in air, the marble will experience very small drag force. Now for energy decay, from class, we know that for an underdamped oscillator, $x(t) \sim e^{-\gamma t}\cos(\omega t + \phi)$. So the energy will decay as $E \propto e^{-\gamma t}$. This loss is very small when γ is small. On the other hand, for large damping, as we saw above $E \propto e^{-2\omega^2/\gamma}$ which will again be small in the case of large damping.

3.

1. Given that there is no damping and the oscillator is being driven externally, the equation describing this is

$$x'' + \omega_0^2 x = f\cos\omega t \tag{26}$$

We want to find the steady state solution which refers to the particular solution. We will use the same approach as in the first question. We note that $\cos\omega t = \text{Re}(e^{i\omega t})$. Let $x_p = Ae^{i\omega t}$. This will give us

$$
\begin{aligned}
A(-\omega^2 + \omega_0^2) &= f \\
A &= \frac{f}{\omega_0^2 - \omega^2}
\end{aligned}
\tag{27}
$$

We are told that the driving frequency is very close to the natural frequency ω_0 i.e. $\omega = (1-\epsilon)\omega_0$ with $0 < \epsilon << 1$. So

$$
\begin{aligned}
\omega_0^2 - \omega^2 &= (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\epsilon\omega_0^2 \\
\implies A &= \frac{f}{2\epsilon\omega_0^2} \quad \text{and} \quad x = \text{Re}\left(\frac{f}{2\epsilon\omega_0^2} e^{i\omega t}\right) = \frac{f}{2\epsilon\omega_0^2}\cos\omega t
\end{aligned}
\tag{28}
$$

So the energy will be

$$
\begin{aligned}
E &= \frac{m}{2}x'^2 + \frac{m}{2}\omega_0^2 x^2 \\
&= \frac{m}{2}\frac{\omega^2 f^2}{4\epsilon\omega_0^4}\sin^2\omega t + \frac{m}{2}\frac{\omega^2 f^2}{4\epsilon\omega_0^4}\cos^2\omega t \\
&= \frac{mf^2}{8\epsilon\omega_0^2}
\end{aligned}
\tag{29}
$$

b. In this case, the oscillator has not reached steady state yet and so we will also need the homogeneous solution. We will again start with the ansatz $x = e^{\alpha t}$

$$
\begin{aligned}
x'' + \omega_0^2 x &= 0 \\
\alpha^2 + \omega_0^2 &= 0 \\
\implies \alpha &= \pm i\omega_0
\end{aligned}
\tag{30}
$$

So we can write the homogeneous solution as $x_h = A\cos\omega t + B\sin\omega t$ and the full solution will be

$$
x = x_h + x_p = A\cos\omega t + B\sin\omega t + \frac{f}{\omega_0^2 - \omega^2}\cos\omega t
\tag{31}
$$

. Using the initial conditions, we get

$$
\begin{aligned}
x(0) &= B + \frac{f}{\omega_0^2 - \omega^2} = 0 \\
x'(0) &= A\omega_0 = 0 \\
\implies A &= 0 \quad B = -\frac{f}{\omega_0^2 - \omega^2}
\end{aligned}
\tag{32}
$$

And so

$$\begin{aligned} x &= \frac{f}{\omega_0^2-\omega^2}\left(\cos\omega t - \cos\omega_0 t\right) \\ &= \frac{2f}{\omega_0^2-\omega^2}\sin\frac{\omega+\omega_0}{2}t\sin\frac{\omega-\omega_0}{2}t \\ &\approx \frac{-f}{2\epsilon\omega_0^2}\sin\omega_0 t\frac{\omega_0\epsilon t}{2} \\ &= \frac{f}{2\omega_0}t\sin\omega_0 t \end{aligned} \tag{33}$$

where we used (28) and the fact that for small x, $\sin x \approx x$. So the energy will be

$$\begin{aligned} E(t) &= \frac{m}{2}x'^2 + \frac{m}{2}\omega_0^2 x^2 \\ &= \frac{mf^2}{8}\left(t^2 + \frac{1}{\omega_0^2}(\sin\omega_0 t)^2 + \frac{t}{\omega_0}\sin 2\omega_0 t\right) \end{aligned} \tag{34}$$

We want to average this over one cycle. The average is defined as

$$\begin{aligned} < E(t) > &= \frac{1}{T}\int_{t-T/2}^{t+T/2} E(t)dt \\ &= \frac{mf^2t^2}{8} + \frac{mf^2}{8}\left(\frac{3+2\pi^2-3\cos 2\omega_0 t}{6\omega_0^2}\right) \end{aligned} \tag{35}$$

where $T = 2\pi/\omega_0$. We will have a quadratic growth in time with an oscillating piece as well but at late times the quadratic part will dominate.

c. At late times, the quadratic in t part will dominate. So if we ignore the oscillating piece, the time that it will take to get to E_{max} will be

$$\begin{aligned} \frac{1}{8}mf^2t^2 &= \frac{mf^2}{8\omega_0^2\epsilon^2} \\ t = \frac{1}{\omega_0\epsilon} &\approx \frac{1}{\omega\epsilon} \end{aligned} \tag{36}$$