Physics 143A: Honors Waves, Optics, and Heat

Spring Quarter 2024
Problem Set #2
Due: 11:59 pm, Thursday, April 3. Please submit to Canvas.

1. **Math** Determine the eigenvalues and eigenvectors of the following matrices (5 points each)
   (a) $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$
   (b) $\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}$
   (c) A particle of mass 1 is moving in a two dimensional plane with potential energy $V(x, y) = x^2 + y^2 - xy - 6x$. Determine the equilibrium position (x_0, y_0), where the potential energy is at the minimum, and the two eigen-frequencies of the particle moving near the potential minimum.
   (Hint: Taylor expand the potential near the minima. You may introduce a new coordinate $u = x - x_0, v = y - y_0$ and show that the equation of motion $m\vec{r}'' = -\nabla V(\vec{r})$ is given by

$$u'' = -2u + v$$
$$v'' = -2v + u.$$

   You can then derive the eigenfrequencies.)
   (d) Derive or simply argue what the eigen-frequencies are if the particle is moving near the minimum of the potential $V(x, y) = e^{x^2+y^2-xy}$.

2. **Damping of coupled oscillators** (10 points each)
   What will happen if you couple two over-damped oscillators? In HW1 problem 1 we solved $x'' + 4x' + 3x = 0$, which describes an overdamped oscillator. Now we couple two of them as

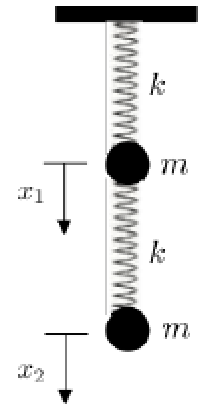
$$x'' + 4x' + 3x = 2y$$
$$y'' + 4y' + 3y = 2x$$

   (a) Write the equations in the vector-matrix form as $\vec{x}''(t) + \hat{\gamma}\vec{x}'(t) + \widehat{M}\vec{x}(t) = 0$, where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and determine the matrices $\hat{\gamma}$ and $\widehat{M}$.
   (b) Determine the four eigen-frequencies and show that some modes can be become underdamped.
   (c) Now consider two identical, critical damped oscillators coupled to each other, namely,

$$x'' + \gamma x' + \omega^2 x = \epsilon y$$
$$y'' + \gamma y' + \omega^2 y = \epsilon x,$$

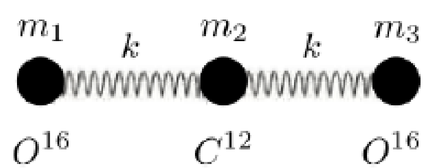
   where the damping is $\gamma = 2\omega$. Show that independent of coupling strength ϵ, 2 eigenmodes become overdamped and 2 eigenmodes become underdamped.

3. **Two masses on two vertical springs** (10 points each)
Two identical masses are attached to two massless springs as shown. Considering only motion in the vertical direction, solve for the motion of each mass about their equilibrium positions.



(a) Given the gravitational pull of *mg* and assuming no friction, determine the displacement of the masses in equilibrium. Write down the differential equations that describe the deviations of the masses x_1 and x_2 from the equilibrium position, see Figure. Show that gravitational pull does not explicitly show up in the equations.
(b) Determine and describe the eigenmodes and their frequencies. (Hint: Define $\omega_0 = \sqrt{k/m}$.)
(c) At $t=0$, the lower mass was slightly displaced by $x_2(0) = D$, while ~~while~~ $x_1(0)= x_1'(0)= x_2'(0)=0$ remain unchanged. Determine their subsequent motion $x_1(t)$ and $x_2(t)$.

4. **Vibrations of a CO2 molecule (10 points each)**
A CO_2 molecule can be modeled as a central carbon atom with mass m_2 connected by 2 identical springs with spring constant k to two oxygen atoms with mass $m_1 = m_3$.



(A) First we consider linear motion of the 3 atoms along the molecular axis. Write down the differential equations that describe their motion x_1, x_2 and x_3 and determine the eigenfrequencies.
(B) Describe the eigenmodes. In principle there should be 3 eigenmodes for 3 oscillators. Explain why you only get two normal modes? Where is the third mode?