PHYS 143 – Problem Set 2

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1.

a. To find the eigenvalues and eigenvectors, we need to solve the following

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \vec{v} = \lambda \vec{v} \tag{1}$$

To get a non-trivial solution, we would want

$$\begin{aligned} \implies \det \begin{pmatrix} A-\lambda & -B \\ B & A-\lambda \end{pmatrix} &= (A-\lambda)^2 + B^2 = 0 \\ \lambda = A \pm iB \end{aligned} \tag{2}$$

To find the eigenvectors, we solve the equation $(M - \lambda I)\vec{v}_\pm = 0$ assuming $\vec{v}_\pm = \begin{pmatrix} c_\pm \\ d_\pm \end{pmatrix}$.

So we have,

$$\begin{pmatrix} A-(A \pm iB) & -B \\ B & A-(A \pm iB) \end{pmatrix} \begin{pmatrix} c_\pm \\ d_\pm \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

Solving these equations will give us

$$\vec{v}_\pm = \begin{pmatrix} \pm i \\ 1 \end{pmatrix} \tag{4}$$

b. Similarly for the second part, we are given

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix} \vec{v} = \lambda \vec{v} \tag{5}$$

Solving for the eigenvalues first i.e. $\det(M - \lambda I) = 0$, we need to solve the equation

$$\begin{aligned} &2\lambda - 2\lambda^2 - \lambda^3 = 0 \\ &\lambda = 0,\ -1 \pm \sqrt{3} \end{aligned} \tag{6}$$

Next, we want to solve for the eigenvectors corresponding to each of these eigenvalues. Let us start with $\lambda_0 = 0$

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix} \vec{v} = 0 \implies \vec{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{7}$$

Similarly for the other two eigenvectors, we get

$$\begin{pmatrix} \pm\sqrt{3} & 1 & -1 \\ 1 & 1 \pm \sqrt{3} & 1 \\ -1 & 1 & \pm\sqrt{3} \end{pmatrix} \vec{v} = 0 \implies \vec{v}_\pm = \begin{pmatrix} 1 \\ 1 \pm \sqrt{3} \\ 1 \end{pmatrix} \tag{8}$$

c. To get the equilibrium position, we need to minimize the potential first. We want

$$\begin{aligned} \frac{\partial V}{\partial x} &= 2x - y - 6 = 0 \\ \frac{\partial V}{\partial y} &= 2y - x = 0 \\ \implies x_0 &= 4,\ y_0 = 2 \end{aligned} \tag{9}$$

Now using the hint, we introduce coordinates $u = x - x_0 = x - 4$ and $v = y - y_0 = y - 2$. In these coordinates, the potential is

$$\begin{aligned} V(u, v) &= (u+4)^2 + (v+2)^2 - (u+4)(v+2) - 6(u+4) \\ &= u^2 + v^2 - uv - 12 \end{aligned} \tag{10}$$

Now we can write the equation of motion as

$$\begin{aligned} & m\ddot{\vec{x}} = -\nabla V \\ & \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} u'' \\ v'' \end{pmatrix} = \begin{pmatrix} -\partial_u V \\ -\partial_v V \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned} \tag{11}$$

So we have a pair of coupled differential equations. Let us pick the ansatz $\vec{x} = e^{\alpha t}\vec{x_0}$, so the equation will become an eigenvalue problem

$$\begin{aligned} & M\vec{x_0} = m\alpha^2 \vec{x_0} \\ & \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} m\alpha^2 & 0 \\ 0 & m\alpha^2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \\ \implies & \begin{pmatrix} -2 - m\alpha^2 & 1 \\ 1 & -2 - m\alpha^2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0 \end{aligned} \tag{12}$$

Diagonalizing the matrix gives us

$$\begin{aligned} m\alpha^2 &= -3, \quad -1 \\ \alpha &= \pm i\sqrt{3} \quad \pm i \end{aligned} \tag{13}$$

where we use the fact that $m = 1$. So we have then for $\alpha = \pm i$,

$$\begin{aligned} \vec{x} &= \vec{A}e^{it} + \vec{B}e^{-it} \\ &= \vec{C}\cos t + \vec{D}\sin t \end{aligned} \tag{14}$$

And similarly for $\lambda = \pm i\sqrt{3}$. So the eigenfrequencies are $\omega = 1,\ \sqrt{3}$.

d. Minimizing the potential $V(x, y) = e^{x^2+y^2-xy}$, we get the equations

$$\begin{aligned} \partial_x V &= (2x - y)V = 0 \\ \partial_y V &= (2y - x)V = 0 \\ \implies x &= 0,\ y = 0 \end{aligned} \tag{15}$$

Now the equation of motion will be

$$\vec{x}'' = \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -\partial_x V \\ -\partial_y V \end{pmatrix} = V \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{16}$$

We can expand the potential around the minima $(x, y) = (0, 0)$. Taylor series expansion for a multivariable function around a point (x_0, y_0) takes the form

$$\begin{aligned} V(x, y) &= V(x_0, y) + V'_x(x_0, y)(x - x_0) + V''_x(x_0)(x - x_0)^2 + \cdots \\ &= V(x_0, y_0) + V'_x(x_0, y_0)(x - x_0) + V'_y(x_0, y_0)(y - y_0) + V''_{xy}(x_0, y_0)(x - x_0)(y - y_0) \\ &\quad + \frac{1}{2}V''_x(x_0, y_0)(x - x_0)^2 + \frac{1}{2}V''_y(x_0, y_0)(y - y)^2 + \cdots \end{aligned} \tag{17}$$

where $V'_x = \partial V/\partial x$ and so on. Using this, we get

$$V(x, y) = 1 - (x^2 + y^2 - xy) \tag{18}$$

Substituting this into the equation of motion, we see that we have at leading order

$$\vec{x}'' = \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -\partial_x V \\ -\partial_y V \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{19}$$

This is exactly the same as the last part now and we know the eigenfrequencies $\omega = 1,\ \sqrt{3}$.

An intuitive way to see why the eigen frequencies is the same is to note that the potential e^V is the locally the same as V near $(x, y) = (0, 0)$ and so the physics will be the same near that point.

2.

a. The equation in matrix form can be written as

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{20}$$

So $\hat{\gamma} = 4\text{I}$ and $\hat{M} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$.

b. Let us assume that $\vec{x} = e^{\alpha t}\vec{A}$ where $\vec{A} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. That will give us

$$\begin{pmatrix} \alpha^2 + 4\alpha + 3 & -2 \\ -2 & \alpha^2 + 4\alpha + 3 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{21}$$

We see that to get a non-trivial solution, we would want the determinant of the matrix to be zero. That gives us

$$\begin{aligned} &\det \begin{pmatrix} \alpha^2 + 4\alpha + 3 & -2 \\ -2 & \alpha^2 + 4\alpha + 3 \end{pmatrix} = 0 \\ &(\alpha^2 + 4\alpha + 3)^2 - 4 = 0 \\ &\alpha = -2 \pm i, \, -2 \pm \sqrt{3} \end{aligned} \tag{22}$$

The real solutions lead to overdamped modes and the imaginary parts lead to underdamped modes- there is still some oscillatory piece left. It turns out that the general solution then takes the form

$$x(t) = e^{-2t}\left((c_1 \cos\omega t + c_2 \sin\omega t)\,\lambda_+ + \left(c_3 e^{\sqrt{3t}} + c_4 e^{-\sqrt{3t}} \right) \lambda_- \right) \tag{23}$$

where we can see that we have an oscillatory piece corresponding and a decaying piece (overdamped mode).

c. Now we can write the equation again in matrix form. The equations are

$$\begin{aligned} x'' + \gamma_1 x' + \omega_1^2 x = \epsilon y \\ y'' + \gamma_2 y' + \omega_2^2 y = \epsilon x \end{aligned} \tag{24}$$

This in matrix form is

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} + \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{25}$$

Assuming $\vec{x} = e^{\alpha t}\vec{x}_0$, we get

$$\begin{pmatrix} \alpha^2 + \alpha\gamma_1 + \omega_1^2 & -\epsilon \\ -\epsilon & \alpha^2 + \alpha\gamma_2 + \omega_2^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{26}$$

where $\gamma_i = 2\gamma_i$. Now to get a non-trivial solution, we would want the determinant of the matrix to be zero. That can then give us the values for α. We get

$$\begin{aligned} (\alpha^2 + \alpha\gamma_1 + \omega_1^2)(\alpha^2 + \alpha\gamma_2 + \omega_2^2) - \epsilon^2 = 0 \\ \alpha = -\frac{1}{2}\left((\omega_1 + \omega_2) \pm \sqrt{(\omega_1 - \omega_2)^2 \pm 4\epsilon}\right) \end{aligned} \tag{27}$$

Now we see that as long as $(\omega_1 - \omega_2)^2 \leq 4\epsilon$, we will always have both real and imaginary solutions for α because of the $\sqrt{}$ term. The case where the frequencies are real i.e. when $\omega_1 = \omega_2$ is a special case of this.

3.

a. Let us set the origin to be the point where the springs hang from i.e. the top wall and let the distance from the origin be denoted by X_i. Now to write the equation of motion for the two masses we note that we have two different kinds of forces acting- one is the gravitational pull downwards and the other is the force from the spring. The equation of motion for the two masses would be

$$\begin{aligned} mX_1'' &= -kX_1 + mg - k(X_1 - X_2) = -k(2X_1 - X_2) + mg \\ mX_2'' &= -k(X_2 - X_1) + mg = -k(X_2 - X_1) + mg \end{aligned} \tag{28}$$

At equilibrium, the forces will balance each other out and the masses will not move. Let us denote the equilibrium positions by X_i^0, then the equations will be

$$\begin{aligned} &- k(2X_1 - X_2^0) + mg = 0 \implies mg = k(2X_1^0 - X_2^0) \\ &- k(X_2^0 - X_1^0) + mg = 0 \implies mg = k(X_2^0 - X_1^0) \end{aligned} \tag{29}$$

Solving these equations, we get

$$\begin{aligned} X_1^0 &= \frac{2mg}{k} \\ X_2^0 &= \frac{3mg}{k} \end{aligned} \tag{30}$$

Let us denote the deviation from the equilibrium position by x_i, so then we have

$$\begin{aligned} x_1 &= X_1 - X_1^0 \\ x_2 &= X_2 - X_2^0 \end{aligned} \tag{31}$$

Now we can re-express the equations of motion (28) above in terms of x_i by writing $X_i = x_i + X_i^0$ and then use (30). We then get

$$\begin{aligned} mx_1'' &= -k(x_1 + \frac{2mg}{k}) + mg - k(x_1 - x_2 + \frac{mg}{k}) = -k(2x_1 - x_2) \\ mx_2'' &= -k(x_2 - x_1) \end{aligned} \tag{32}$$

we see that the gravitational pull does not explicitly show up in the equations.

b. Now to solve the equations above, let us assume again $\vec{x} = e^{\alpha t}\vec{x_0}$. So the equation in matrix form can be written as be

$$\alpha^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \omega_0^2 \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \tag{33}$$

We can then solve for the eigenvalues and eigenvectors now. For the eigen values, we get

$$\begin{aligned} &(\alpha^2 + 2\omega_0^2)(\alpha^2 + \omega_0^2) - \omega_0^4 = 0 \\ &\alpha_\pm^2 = -\omega_0^2 \left(\frac{3 \pm \sqrt{5}}{2} \right) \end{aligned} \tag{34}$$

with eigenvectors

$$\lambda_\pm = \frac{1}{2} \begin{pmatrix} -1 \pm \sqrt{5} \\ 2 \end{pmatrix} \tag{35}$$

So the general solution will have the form

$$\vec{x}(t) = \vec{\lambda}_+ \left(A_1 \cos\omega_+ t + B_1 \sin\omega_+ t \right) + \vec{\lambda}_- \left(A_2 \cos\omega_- t + B_2 \sin\omega_- t \right) \tag{36}$$

where $\omega_\pm = \frac{\omega_0}{2}(\sqrt{5} \pm 1)$.

c. Next, we impose the initial conditions. We are given that $x_1(0) = 0$ and $x_2(0) = D$. In other words, we have

$$\vec{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \frac{A_1}{2} \begin{pmatrix} -1 + \sqrt{5} \\ 2 \end{pmatrix} + \frac{A_2}{2} \begin{pmatrix} -1 - \sqrt{5} \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ D \end{pmatrix} \tag{37}$$

Solving these two equations

$$\begin{aligned} &\frac{1}{2}(-1 + \sqrt{5})A_1 + \frac{1}{2}(-1 - \sqrt{5})A_2 = 0 \\ &A_1 + A_2 = D \\ \implies &A_1 = \frac{1}{10}\left(5 + \sqrt{5}\right) D, \; A_2 = \frac{1}{10}\left(5 - \sqrt{5}\right) D \end{aligned} \tag{38}$$

Imposing the second condition $x_1'(0) = x_2'(0) = 0$ gives $B_1 = B_2 = 0$. So we will finally have

$$x(t) = \frac{D}{20}(5+\sqrt{5})\begin{pmatrix} -1+\sqrt{5} \\ 2 \end{pmatrix}\cos\omega_+ t + \frac{D}{20}(5-\sqrt{5})\begin{pmatrix} -1-\sqrt{5} \\ 2 \end{pmatrix}\cos\omega_- t \tag{39}$$

4.

- The equation of motion for the three molecules can be directly read by looking at how the forces are acting. An alternate way is to consider the total energy and derive the force equation from there. So, we have

$$U(x_1, x_2, x_3) = \frac{1}{2}k(x_3 - x_2)^2 + \frac{1}{2}k(x_2 - x_1)^2 \tag{40}$$

And so

$$\begin{aligned} F_1 &= m x_1'' = -\frac{\partial U}{\partial x_1} = -k(x_1 - x_2) \\ F_2 &= m x_2'' = -\frac{\partial U}{\partial x_2} = -k(x_2 - x_3) - k(x_2 - x_1) \\ F_3 &= m x_3'' = -\frac{\partial U}{\partial x_3} = -k(x_3 - x_2) \end{aligned} \tag{41}$$

In matrix form, this would be (we again write $\vec{x}(t) = e^{\alpha t}\vec{x}_0$)

$$\alpha^2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\omega^2 \begin{pmatrix} 1 & -1 & 0 \\ -\beta & 2\beta & -\beta \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{42}$$

Solving for the eigenvalues and eigenvectors, we get

$$\alpha_\pm = \pm i\omega_0,\ \alpha_0 = 0,\ \alpha_\pm^\beta = \pm\omega\sqrt{2\beta+1} \tag{43}$$

with corresponding eigenvectors

$$\lambda_\pm = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},\ \lambda_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \lambda_\beta = \begin{pmatrix} 1 \\ -2\beta \\ 1 \end{pmatrix} \tag{44}$$

c. The mode corresponding to λ_0 is just pure translation.