Physics 143A: Honors Waves, Optics, and Thermo

Spring Quarter 2024

Problem Set #3

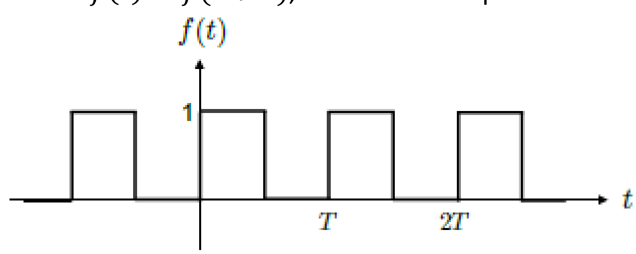
Due: 11:59 pm, Wednesday, April 10. Please submit to Canvas.

1. **(Math) Fourier expansion and Fourier transform exercise (6 points each)**
   You may Fourier expand a periodic function or Fourier transform a general function as

   Fourier series expansion: $y(x) = f(x+L) \equiv a_0 + \sum_{n=1} a_n \cos\frac{2\pi n x}{L} + \sum_{n=1} b_n \sin\frac{2\pi n x}{L}$
   Fourier transform: $y(x) = \int y(k) e^{ikx} dk$

   Determine the Fourier series of the following functions
   (a) Fourier expand $y(x) = |\sin x|$
   Hint: first determine the period of the function and then the basis you need to expand it.
   (b) Fourier expand an infinite series of periodic impulses
   $y(x) = \sum_{n\in\mathbb{Z}} \delta(x-n)$, where $\mathbb{Z} = \{0, \pm 1, \pm 2 \ldots\}$ includes all integers and $\delta(x)$ is Dirac's Delta function.
   (c) Fourier transform a single impulse $y(x) = \delta(x)$
   (d) Fourier transform a series of random impulses $y(x) = \sum_n \delta\,(x - x_n)$.
   (e) Show that given $f(x) = \int f(k) e^{ikx} dk$, we have the following normalization condition
   $$\int f^*(x) f(x) dx = 2\pi \int f^*(k) f(k) dk$$
   Hint: Use the formula $\int e^{ikx} dx = 2\pi\delta(k)$

2. **Fourier series and transforms of a square wave** (7 points each)
   Consider a periodic function $f(t) = f(t+T)$, where T is the period. See below



   We will expand it in 3 ways and compare the results.
   (a) Expand $f(t)$ with the trigonometric functions with the same period
   $$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos\frac{2\pi n t}{T} + b_n \sin\frac{2\pi n t}{T}).$$
   Determine a_0, a_n and b_n.
   (b) Expand $f(t)$ with the exponential series with the same period

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi nt}{T}}.$$

Determine c_n.

(c) Fourier transform $f(t)$ over the entire range

$$f(t) = \int \tilde{f}(\omega) e^{i\omega t} d\omega.$$

Determine $\tilde{f}(\omega)$.

(d) All these expansions are equivalent. Show that c_n and $\tilde{f}(\omega)$ can be expressed in terms of a_0, a_n and b_n.

3. General solution of a driven harmonic oscillator (9 points each)

(a) A driven oscillator is described by equation of motion:

$$x''(t) + \gamma x'(t) + \omega_0^2 x(t) = f(t)$$

Show that the particular solution is

$$x(t) = \frac{1}{2\pi} \int \int \frac{f(\tau) e^{i\omega(t-\tau)}}{\omega_0^2 - \omega^2 + i\gamma\omega} d\omega d\tau.$$

(b) As an example, we consider the external force $f(t)$ as described in question 2. Determine the explicit form of the solution of the oscillator.

Hint: $x(t) = \frac{1}{2\omega_0^2} + \frac{1}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} Im[\frac{e^{i\omega_n t}}{\omega_0^2 - \omega_n^2 + i\gamma\omega_n}]$, where $\omega_n = \frac{2\pi n}{T}$.

4. Damped wave equation and guitar string (8 points each)

A guitar string moves according to the following damped wave equation

$$\rho \partial_t^2 \psi(x,t) + b \partial_t \psi(x,t) = E \partial_x^2 \psi(x,t)$$

with the fixed boundary conditions

$$\psi(0,t) = \psi(L,t) = 0$$

(a) Determine the dispersion $\omega(k)$ of the eigenmodes, where ω is the angular frequency of the wave and k is the wavenumber.

(Hint: Dispersion is the relation between angular frequency and wave number. You may use the ansatz $\psi = Ae^{ikx} e^{i\tilde{\omega} t}$ and show that the eigenfrequency is $\omega = Re[\tilde{\omega}] = \sqrt{\frac{E}{\rho} k^2 - \frac{b^2}{4\rho^2}}$.

Next show that to satisfy the boundary condition the wavefunction should have the form $\psi = A \sin kx \, e^{i\omega t}$ and the wavenumber $k = k_n = \frac{(n+1)\pi}{L}$ can only take discrete values with n=0,1,2,3….)

(b) Assume the initial position and velocity of the string are given by

$$\psi(x,0) = f(x), \qquad \partial_t \psi(x,0) = g(x)$$

Consider a frictionless string $b = 0$ for simplicity, determine the wavefunction $\psi(x,t)$ for $t > 0$ and verify that the solution is always real.

(c) A string instrument can produce a rich spectrum with fundamental ω_0 and unique overtones $\omega_n, n = 1, 2$ .... Without dissipation $b = 0$ the pitch of the overtone is $\omega_n = (n + 1)\omega_0$. Show that in the presence of small damping $0 < b \ll E\rho/L^2$, the overtones are detuned from the fundamental as $\omega_n = (n+1)\omega_0 + \delta_n$. For small damping, show that

$$\delta_n \approx (n + 1 - \frac{1}{n+1}) \frac{b^2}{8\rho^2\omega_0}$$

Such effect is more severe for in the range of low frequency.
(Hint: in the presence of damping, the fundamental frequency is also shifted.)