PHYS 143 – Problem Set 4

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1. We want to work with $f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$. So

$$\begin{aligned} \delta(x) &= \lim_{\Delta \to 0} \frac{1}{\Delta} f\left(\frac{x}{\Delta}\right) \\ &= \lim_{\Delta \to 0} \frac{1}{\Delta} \exp\left(\frac{-x^2}{\Delta^2}\right) \end{aligned} \tag{1}$$

1a. Let us use $t = 1/\Delta$. So

$$\begin{aligned} \delta(x \neq 0) &= \lim_{\Delta \to 0} \frac{1}{\Delta} \exp\left(\frac{-x^2}{\Delta^2}\right) \\ &= \lim_{t \to \infty} t \exp\left(-x^2 t^2\right) \\ &= \lim_{t \to \infty} \frac{1}{2t \exp\left(-x^2 t^2\right)} \\ &= 0 \end{aligned} \tag{2}$$

where in the third line we used L'Hôpital's rule $\lim_{x \to x_0} = \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$.

1b. We can see from the definition that $\delta(x = 0) = \lim_{\Delta \to 0} \frac{1}{\Delta}$ diverges.

1c. We have

$$\begin{aligned} \int \delta(x) dx &= \lim_{\Delta \to 0} \frac{1}{\Delta \sqrt{\pi}} \int \exp\left(-x^2/\Delta^2\right) dx \\ &= \lim_{\Delta \to 0} \frac{1}{\Delta \sqrt{\pi}} \sqrt{\pi} \Delta = 1 \end{aligned} \tag{3}$$

1d. Consider the RHS. We have

$$\begin{aligned} \int f(u) \delta(x - u) du &= \lim_{\Delta \to 0} \int \frac{1}{\sqrt{\pi} \Delta} f(u) \exp\left(-(x-u)^2/\Delta^2\right) du \\ &= \lim_{\Delta \to 0} \int \frac{\Delta}{\sqrt{\pi} \Delta} f(\Delta y + x) \exp\left(-y^2\right) dy \\ &= \frac{1}{\sqrt{\pi}} \int f(x) \exp\left(-y^2\right) dy = f(x) \end{aligned} \tag{4}$$

In the second line, we make a change of variables to $y = \frac{u-x}{\Delta} \implies u = \Delta y + x$.

b1. Let us make a change of variables to $y = ax + b$. So then we have

$$\int dx\, g(x)\delta(ax+b) = \frac{1}{a}\int dy\, g\left(\frac{y-b}{a}\right)\delta(y) = \frac{1}{a}g\left(-\frac{b}{a}\right) \tag{5}$$

b2. For this part, we can integrate by parts. We have

$$\begin{aligned}\int g(x)\frac{d\delta(x)}{dx}\, dx &= \int dx\, \left[\frac{d}{dx}\left(g(x)\delta(x)\right) - g'(x)\delta(x)\right] \\ &= \left(g(x)\delta(x)\right)\big|_{-\infty}^{\infty} - \int dx\, g'(x)\delta(x) \\ &= -g'(0)\end{aligned} \tag{6}$$

2.

a. To understand the region around $t = 0$, let us integrate on a small region around that. So we integrate the equation from $0 - \epsilon$ to $0 + \epsilon$

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} dt\, \left(x'' + \omega_0^2 x\right) = \lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} dt\, f(t) \\ &\implies x'(0^+) - x'(0^-) = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} dt f(t) \\ &\implies v = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} dt f(t) \\ &\implies f = v\delta(t)\end{aligned} \tag{7}$$

where in the last line, we used the result from **1d**. This means then that we are solving for what is called the Green's function i.e. the solution to the equation $\mathcal{L}G(t;0) = \delta(t)$ where

$$\mathcal{L} = \frac{1}{v}\frac{d^2}{dt^2} + \frac{\omega_0^2}{v}$$

and the usual notation for the Green's function is that the first argument is the variable we are working with and the second argument is the point at which the delta function peaks (here it is $\tau = 0$). One way to solve for the Green's function is to note that the equation is homogeneous except at $t = 0$. So the solution when $t \neq 0$ should be the solution to $\mathcal{L}G(t;0) = 0$ i.e. the solution to

$$\frac{1}{v}\frac{d^2x}{dt^2} + \frac{\omega_0^2}{v}x = 0$$

We know the general solution to this equation. So we say,

$$x_G(t) \equiv G(t;0) = \begin{cases} A\sin(\omega_0 t) + B\cos(\omega_0 t) & t < 0 \\ C\sin(\omega_0 t) + D\cos(\omega_0 t) & t > 0 \end{cases} \tag{8}$$

The constants A, B, C, D will be fixed using the initial conditions and the matching conditions at $t = 0$. We are given that the initial conditions at $t = -\infty$ which then give us that $A = B = 0$

$$\begin{aligned} & x(-\infty) = 0 \,\text{and}\, x'(-\infty) = 0 \\ \implies & A = 0, \quad B = 0 \end{aligned}$$

The second set of conditions are the matching conditions. These come from demanding that the Green's function be continuous at the boundary. We are already given these conditions in the question (we also derived them on our own in the discussion section).

$$\begin{aligned} x_G(0) = 0 &\implies D = 0 \\ x'_G(0) = v &\implies C = \frac{v}{\omega_0} \end{aligned} \tag{9}$$

Imposing this on our solution (8), we get

$$x_G \equiv G(t; 0) = \begin{cases} 0 & t < 0 \\ \frac{v}{\omega_0} \sin(\omega_0 t) & t > 0 \end{cases} \tag{10}$$

2b. The advantage of finding the Green's function for any operator is that then we can find the solution to any general source function. So for $\mathcal{L}x(t) = f(t)$, the solution is

$$x(t) = \int G(t; \tau) f(\tau) d\tau$$

where $LG(t; \tau) = \delta(t - \tau)$. One way to see that is to just act on both sides by $\mathcal{L}$ and check that $x(t)$ it is actually a solution.

Another way would be to write the source function as a summation of different delta functions- this is what we do in this part. So we want to solve $\mathcal{L}x(t) = F(t)$. We can write $F(t)$ as

$$F(t) = \int F(\tau)\delta(t - \tau)\, d\tau$$

For each $F(\tau)\delta(t - \tau)$, we know the solution from **2a.** It will be $\frac{F(\tau)}{v} G(t; \tau)$ i.e.

$$x_\tau = \begin{cases} 0 & t < \tau \\ \frac{F(\tau)}{v} \sin \omega_0 (t - \tau) & t > \tau \end{cases}$$

So the solution to $F(t)$ will be a superposition of x_τ and so

$$x(t) = \int_{-\infty}^{t} d\tau \, x_\tau = \int_{-\infty}^{t} \frac{F(\tau)}{v} \sin \omega_0 (t - \tau) \tag{11}$$

3.

a. The kinetic energy per unit length will be

$$\rho_K = \frac{1}{2}\rho v^2$$

Now we note that in a small region Δx, the velocity will be $v = \partial_t \psi$ and so

$$\rho_K = \frac{1}{2}\rho(\partial_t \psi)^2 \tag{12}$$

To calculate the potential energy, we can treat the medium like a spring- so the first thing we will need will be the extension of the string in a small region Δx.

$$\begin{aligned} \Delta l &= \sqrt{(\Delta y)^2 + (\Delta x)^2} - \Delta x \\ &= \Delta x \left(\sqrt{1 + (\Delta y/\Delta x)^2} - 1 \right) \\ &\simeq \frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2 \Delta x \\ &= \frac{1}{2}(\partial_x \psi)^2 \Delta x \end{aligned} \tag{13}$$

And the restoring force will be

$$F = -k\Delta l = 2T \tag{14}$$

The factor of 2 comes from noting that the string will be pulled from both ends. So the potential energy will be

$$\begin{aligned} \rho_U &= \frac{1}{2}k(\Delta l)^2 = \frac{1}{2}(k\Delta l)(\Delta l) \\ &= \frac{T}{2}(\partial_x \psi)^2 \end{aligned} \tag{15}$$

b. We can use the form of the wave solution given to use and that gives us

$$\begin{aligned} \rho_k &= \frac{1}{2}\rho(\partial_t \psi)^2 = \frac{1}{2}\rho A^2 k^2 v^2 \sin^2 k(x - vt) = \frac{1}{2}TA^2k^2 \sin^2 k(x - vt) \\ \rho_U &= \frac{1}{2}T(\partial_x \psi)^2 = \frac{1}{2}TA^2k^2 \sin^2 k(x - vt) = \rho_k \end{aligned} \tag{16}$$

So we see that the energy travels with the wave in a sinusoidal way with a velocity $v = \sqrt{T/\rho}$. Points where $\rho_K + \rho_U = 0$ will be when

$$\sin k(x - vt) = 0 \implies k(x - vt) = n\pi \tag{17}$$

c. The energy flux is

$$j_E = -\frac{dW}{dt}$$

Now, we know from the hint that

$$
\begin{aligned}
\frac{dW}{dt} &= T_y \frac{d\psi}{dt} \\
&= T_x \partial_x \psi \partial_t \psi \\
&\simeq T \partial_x \psi \partial_t \psi
\end{aligned} \tag{18}
$$

And so $j_E = -T\partial_x\psi\partial_t\psi$.

d. Substituting the wave form in the formula above gives us

$$
\begin{aligned}
j_E &= -T\partial_x\psi\partial_t\psi \\
&= \pm TA^2k^2v\sin^2 k(x \pm vt) = \pm v(\rho_K + \rho_U) \equiv \pm v\rho_E
\end{aligned} \tag{19}
$$