PHYS 143 – Problem Set 4

Instructor: Cheng Chin

1.

a. The divergence of a vector tells us whether it is a source or a sink and the curl tells us about circulations. Let us calculate that for the two vectors.

For $\vec{A}_1 = (y^2 x, 0, -yx^2)$, we see that

$$\begin{aligned}\nabla \cdot \vec{A}_1 &= \partial_x A_x + \partial_y A_y + \partial_z A_z \\ &= y^2 > 0 \implies \text{It is a source}\end{aligned} \tag{1}$$

And the curl

$$\begin{aligned}\nabla \times \vec{A}_1 &= (\partial_y A_z = \partial_z A_x)\hat{x} - (\partial_x A_x - \partial_z A_x)\hat{y} + (\partial_x A_y - \partial_y A_x)\hat{z} \\ &= -x^2\hat{x} + 2xy\hat{y} - 2yz\hat{z} \\ &\equiv (-x^2, 2xy, -2yx) \implies \text{carries circulation}\end{aligned} \tag{2}$$

For $\vec{A}_2 = (\cos z, \sin x - \sin y, -\cos z$. We see that the divergence is

$$\nabla \cdot \vec{A}_2 = -\cos y + \sin z \tag{3}$$

And the curl

$$\nabla \times \vec{A}_2 = (0, -\sin z, \cos x) \tag{4}$$

b. We can use Einstein's summation notation here to simplify things- so repeated indices imply summing over them. In other words $v_i j^i \equiv \sum_i v_i j^i$.

So the first equation can be written as

$$\begin{aligned}\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) &= \partial_i (\phi \partial_i \psi - \psi \partial_i \phi) \\ &= \partial_i \phi \partial_i \psi + \phi \partial_i^2 \psi - \partial_i \psi \partial_i \phi - \psi \partial_i^2 \phi_i \\ &= \phi \partial_i^2 \psi - \psi \partial_i^2 \phi \\ &= \phi \nabla^2 \psi - \psi \nabla^2 \phi\end{aligned} \tag{5}$$

And the second equation can be written as

$$\begin{aligned}
\nabla \times (\nabla \times \vec{A}) &= \epsilon^{ijk}\partial_j(\nabla \times \vec{A})_k \\
&= \epsilon^{ijk}\partial_j\left(\epsilon^{kmn}\partial_m A_n\right) \\
&= \epsilon^{ijk}\epsilon^{kmn}\partial_j\partial_m A_n \\
&= \epsilon^{kij}\epsilon^{kmn}\partial_j\partial_m A_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\,\partial_j\partial_m A_n \\
&= \partial_i\partial_j A_j - \partial_j\partial_j A_i = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}
\end{aligned} \tag{6}$$

2.

a. We need to impose the boundary conditions which in this case lies at $x = 0$

$$\begin{aligned}
\psi_L(0,t) &= \psi_R(0,t) \\
\partial_x\psi_L(0,t) &= \partial_x\psi_R(0,t)
\end{aligned} \tag{7}$$

Since these conditions should hold at all times, we get

$$\begin{aligned}
A + B &= C \\
Ak - Bk &= Ck^*
\end{aligned} \tag{8}$$

Solving these, we get

$$\begin{aligned}
C &= \frac{2A}{(1 + k^*/k)} \\
B &= \frac{C}{2}\left(1 - \frac{k^*}{k}\right) = A\left(\frac{k - k^*}{k + k^*}\right)
\end{aligned} \tag{9}$$

b. Consider now the case with $t \to -t$. We have

$$\psi(x,t) = \begin{cases} Ae^{ik(x - v_R t)} + Be^{ik(-x - v_R t)} & x > 0 \\ Ce^{ik^*(x - v_L t)} & x < 0 \end{cases} \tag{10}$$

Let us treat the two waves (with amplitude B and C) separately like mentioned in the question. Consider the wave with amplitude C as the incident wave: it will have a transmitted part and a reflected part- we will denote their amplitudes with C_T and C_R

$$\psi(x,t) = \begin{cases} C_T e^{ik(x - v_R t)} & x > 0 \\ Ce^{ik^*(x - v_L t) + C_R e^{ik^*(-x - v_L t)}} & x < 0 \end{cases} \tag{11}$$

We will have the same boundary conditions like in the previous part

$$\begin{aligned} C_T &= C + C_R \\ kC_T &= k^*C - k^*C_R \end{aligned} \tag{12}$$

Solving this gives

$$\begin{aligned} C_T &= \frac{2C}{(1+k/k^*)} \\ C_R &= C\left(\frac{k^*-k}{k^*+k}\right) \end{aligned} \tag{13}$$

Similarly, treating the wave with amplitude B independently, it will have a transmitted part B_T and reflected part B_R and so the boundary conditions

$$\begin{aligned} &B + B_R = B_T \\ &-kB + kB_R = -k^*B_T \end{aligned} \tag{14}$$

Solving this, we get

$$\begin{aligned} B_T &= \frac{2B}{1+k^*/k} \\ B_R &= \frac{B(k-k^*)}{(k+k^*)} \end{aligned} \tag{15}$$

And finally the relation between B and C from the boundary conditions is

$$\begin{aligned} A + B &= C \\ Ak - Bk &= Ck^* \end{aligned} \tag{16}$$

giving us

$$B = \frac{C}{2}\left(1 - \frac{k^*}{k}\right) \implies B_T = C\left(\frac{k-k^*}{k+k^*}\right) = -C_R \tag{17}$$

3. Given the values for the pressure corresponding to the sound in decibels, we can check that the relation between the sound level in decibels is related to the pressure as

$$dB = 20\log_{10}\left(\frac{P}{P_0}\right) \quad P_0 = 20\mu Pa \tag{18}$$

a. Let the wave have amplitude A: $\phi(x,t) = A\sin(kx - \omega t)$. In class, you found that

$$\begin{aligned} \Delta P &= -\frac{1}{\beta}\phi'(x) = -\frac{A}{\beta}k\cos(kx-\omega t) \\ \implies A &= \frac{\beta\Delta P}{k} = \frac{\beta v}{\omega}\Delta P \end{aligned} \tag{19}$$

We are given n and β so we can get $v_{\text{air}} = \sqrt{1/n\beta} \simeq 337.4\, ms^{-1}$. From here, we can calculate the amplitude

$$A = \begin{cases} 7.7 \times 10^{-11} m & (0 \text{ dB}) \\ 7.7 \times 10^{-5} m & (100 \text{ dB}) \end{cases} \tag{20}$$

b. We see using our formula relating the sound level to pressure that 200 dB corresponds to $200 = 20 \log\left(\frac{P}{20\times10^{-6}}\right) \implies P = 2 \times 10^5$ Pa. We know that the pressure in air is 10^5 Pa. When a sound wave passes by, there will be a change in the pressure such that the new pressure will be given by

$$P = P_0 - \Delta P = P_0 - \frac{1}{\beta}\phi'(x) \tag{21}$$

This pressure can't be negative. But we see that when the sound level is 200 dB i.e. corresponding to pressure gradient of $\Delta P = 2 \times 10^5$ Pa, $P < 0$ which is not possible. We can in fact calculate the upper bound of sound that can pass by noting that the pressure can't go below 0 so the maximum value of $\Delta P = 10^5$ Pa which corresponds to

$$20 \log\left(\frac{10^5}{20 \times 10^{-6}}\right) \simeq 194 \text{ dB} \tag{22}$$

c. Let us assume using the hint that the sound level that can damage our ear is 100 dB. We see that, that corresponds to a pressure gradient of $\Delta P = 2$ Pa. Last homework, we found that the energy density is given by

$$\begin{aligned} j_E(x,t) &= T\partial_t\phi\partial_x\phi = TA^2\omega\kappa\cos^2(kx-\omega t) \\ \implies \langle j_E\rangle &= \frac{1}{2}TA^2\omega\kappa = T\omega k\frac{\beta^2v^2}{\omega^2}(\Delta P)^2 = (\Delta P)^2\beta v \end{aligned} \tag{23}$$

where we used the formula for $A = \frac{\beta v}{\omega}\Delta P$ and used $v = \sqrt{T/n} = \sqrt{1/\beta n} \implies T = 1/\beta$. Using the numerical values from the table, we see

$$\langle j_E\rangle = \frac{1}{2}(\Delta P)^2\beta v = 4.86 \times 10^{-3}\, W/m^2 \tag{24}$$

d. We saw from the last part that $j_E \sim (\Delta P)^2\beta = \frac{A^2}{\beta v}\omega^2$. So we get

$$A = \frac{\sqrt{j_E}}{\omega}\left(\frac{\beta}{n}\right)^{1/4} \tag{25}$$

So for fixed j_E and ω, we see that the maximum displacement will be in air.

4.

a. We start with the third Maxwell's equation and use the result from 1b

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 E \\
\implies -\vec{\nabla} \times (\partial_t \vec{B}) &= \frac{1}{\epsilon}\vec{\nabla}\rho - \nabla^2 E \\
\implies \partial_t \left(\vec{\nabla} \times \vec{B}\right) &= \nabla^2 E - \frac{1}{\epsilon}\vec{\nabla}\rho \\
\implies \mu\epsilon\partial_t^2 E + \mu\sigma\partial_t E &= \nabla^2 E - \frac{1}{\epsilon}\vec{\nabla}\rho
\end{aligned} \tag{26}$$

where in the last line we used, $\vec{j} = \sigma\vec{E}$. Similarly for the $\vec{B}$,

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} \\
\implies \mu\epsilon\partial_t(\vec{\nabla} \times \vec{E}) + \mu\sigma(\vec{\nabla} \times \vec{E}) &= 0 - \nabla^2 \vec{B} \\
\mu\epsilon\partial_t^2 \vec{B} + \mu\sigma\partial_t \vec{B} &= \nabla^2 \vec{B}
\end{aligned} \tag{27}$$

b. We want to solve the wave equation for the electric field when $\rho = 0$

$$\mu\epsilon\partial_t^2 E + \mu\sigma\partial_t E = \nabla^2 E$$

Let us use the ansatz $\vec{E} = \vec{E}_0 e^{i(\tilde{k}z - \omega t)}$. Substituting this in the equation, we get

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \tag{28}$$

So we see that $\tilde{k}$ has an imaginary part also implying that there will be a decaying part of the electric field as well in addition to the usual oscillatory piece. So writing $\tilde{k} = k + \frac{i}{z_0} \implies \tilde{k}^2 = k^2 - \frac{1}{z_0^2} + 2i\frac{k}{z_0}$. So comparing the real and imaginary parts with the equation for $\tilde{k}^2$ above, we get

$$\begin{aligned}
\frac{2k}{z_0} &= \mu\sigma\omega \\
\frac{1}{z_0} &= \frac{\mu\sigma\omega}{2k}
\end{aligned} \tag{29}$$

And

$$\begin{aligned}
k^2 - \frac{1}{z_0^2} &= \mu\epsilon\omega^2 \\
k^2 - \left(\frac{\mu\sigma\omega}{2k}\right)^2 &= \mu\epsilon\omega^2 \\
k^4 - \left(\frac{\mu\sigma\omega}{2}\right)^2 - (\mu\epsilon\omega^2)k^2 &= 0 \\
k^2 &= 2\mu\epsilon\omega^2 \left(1 \pm \left(1 + \frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/2}\right)
\end{aligned} \tag{30}$$

And so we get

$$v_p = \frac{\omega}{k} = \sqrt{\frac{2}{\mu\epsilon\left(1 + \sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}}\right)}} \tag{31}$$

And

$$z_0 = \frac{2k}{\mu\sigma\omega} = \frac{2}{\mu\sigma v_p} \tag{32}$$