Physics 143A: Honors Waves, Optics, and Heat

Spring Quarter 2024

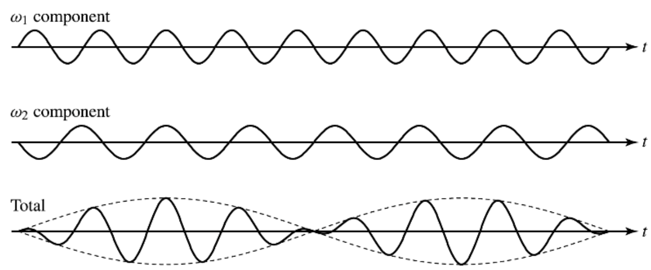
Problem Set #6

Due: 11:59 pm, Wednesday, May 1. Please submit to Canvas.

1. Propagation of a wavepacket (10 points each)

Given the dispersion $\omega(k)$ of a medium in which wave propagates we will clarify the difference between phase velocity $v_p = \omega/k$ and group velocity $v_g = d\omega/dk$ with examples.

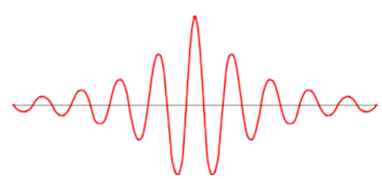
a) Beats of two sinusoidal waves



Consider the superposition of two waves $\psi = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$, where $\omega_1 = \omega(k_1)$ and $\omega_2 = \omega(k_2)$.

Show that the wavefunction can also be described by an infinite traveling wave that propagates at the speed $v_p = (\omega_1 + \omega_2)/(k_1 + k_2)$ with an envelope function (dashed line in the figure) that propagates at the speed $v_g = (\omega_1 - \omega_2)/(k_1 - k_2)$. When the wavenumbers are similar $k_1 \approx k_2$ the velocity v_p becomes the phase velocity and v_g becomes the group velocity.

b) Sinusoidal waves with a generic envelop function



We model a pulse, see right figure, by the product of an infinite traveling wave $e^{i(k_0 x - \omega_0 t)}$, where $\omega_0 = \omega(k_0)$ and a generic envelop function $A(x)$.

Given the initial condition

$$\psi(x, t = 0) = A(x) e^{ik_0 x}$$

prove that to leading order $\omega(k) \approx \omega(k_0) + (k - k_0)\omega'(k_0)$, the wavefunction is

$$\psi(x, t) = A\left(x - v_g t\right) e^{ik_0 (x - v_p t)},$$

where phase velocity is $v_p = \frac{\omega_0}{k_0}$ and $v_g = \frac{d\omega(k_0)}{dk}$.

(Hint: You may write the generic solution of the wave equation in the Fourier space as $\psi(x, t) = \int f(k) e^{i[kx - \omega(k)t]} dk$, where you can determine $f(k)$ from the initial condition.)

c) A musician is playing in a concert hall and you are sitting $L = 100$ ft from the stage. The air is weakly dispersive $\omega(k) \approx v_p k + \epsilon k^2$ with $\epsilon = 8 \times 10^{-4}\ m^2/s$. When the pianist plays the lowest key A_0 at $\omega_L = 2\pi \times 27.5$Hz and the highest key C_8 at $\omega_R = 2\pi \times 4186$ Hz simultaneously which key will reach your ear first and how long is the delay between the 2 notes?
(Hint: you may use the nominal sound speed in air is 343 m/s)

2. Doppler effect and echo (20 points)

Old-fashioned Doppler radar guns use ultrasound to measure the velocity of an object. (Modern ones use microwaves). Assume the gun emits sound wave at frequency ω, the phase velocity is v_p and the police detects the frequency of the reflected waves as ω_r. From the difference, the police determine the speed of the car.

a) A car is moving straight away from the police at the speed $v < v_p$, show that the frequency of the reflected waves has a frequency of

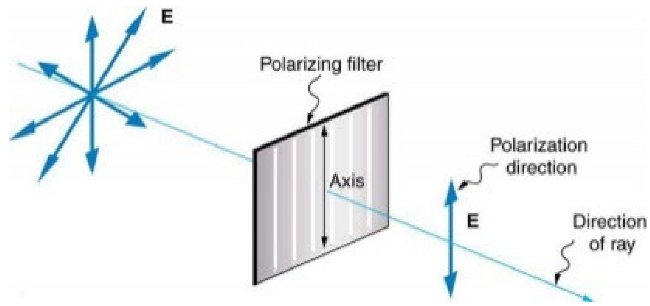
$$\omega_r = \omega_0 \frac{v_p - v}{v_p + v}$$

b) The above formula is inaccurate in the presence of strong wind. Assume the wind is in the same direction as the car and the wind speed is w, show that the frequency of the reflected wave is

$$\omega_\mathrm{r} = \omega_0 \frac{v_p - w}{v_p + w} \frac{v_p + w - v}{v_p - w + v}$$

Hint: Sound propagates in air. So when the air moves at velocity w, sound propagates faster in the same direction at velocity $v_p + w$, and slower in the opposite direction at $v_p - w$. Determine the time duration when two consecutive wavefronts are reflected by the car, and the spatial separation between them. The frequency of the echo here is determined by the duration the police receives two consecutive reflected wavefronts from the car.

3. Polarizers (10 points each)
   Polaroid polarizing filters are made of nitrocellulose polynemer film, where the polymers form needle-like crystals that only absorb light with polarization (direction of electric field) along the direction of the crystals. The axis of the polarizer is typically defined as the direction that the light can transmit. See figure, where the polarizer axis is in the vertical (x-)direction, and the light propagates in z.



   a) Assume the incident beam electric field is given by $E_i = (E_x, E_y)e^{ik(z-ct)}$, and the polarizer is aligned in the x direction as shown in the figure. The transmission rejects E field in the y-direction and thus the transmitted beam is $E_t = E_x(1{,}0)e^{ik(z-ct)}$. If we place a second polarizer rotated by 90 degrees relative to the first one, show or argue that the transmission is zero.

   b) Now we add a third polarizers between the two polarizers at an angle θ relative to the first one. Show that light can transmit again with electric field $E_i = E_x(0, \sin\theta\cos\theta)$.

   c) Show that when an incident beam with electric field $E_{in} = (E_x^{in}, E_y^{in})$ passes through a polarizer rotated by $+\theta$ relative to the x-axis, the outgoing field is given by

$$\begin{pmatrix} E_x^{out} \\ E_y^{out} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{pmatrix} E_x^{in} \\ E_y^{in} \end{pmatrix}.$$

   (Hint: You may use superposition principle to determine the elements of the polarizer matrix.)