Physics 143A: Honors Waves, Optics, and Heat
Spring Quarter 2024
MidTerm
April 8, 2024

Physics

- Newton's law: $m\vec{x}''(t) = \vec{F}(t)$
- Hooke' law: $\vec{F}(t) = -k\vec{x}(t)$
- Stokes' drag: $\vec{F}(t) = -b\vec{x}'(t)$

Math

- 2nd order ordinary differential equation
  $$ax''(t) + bx'(t) + cx(t) = f(t)$$
- Characteristic equation $a\alpha^2 + b\alpha + c\alpha = 0$
- Solution of the quadratic equation $\alpha = -\frac{b \pm \sqrt{b^2-4ac}}{2a}$
- Euler's formula $e^{i\phi} = \cos\phi + i\sin\phi$
- Matrix determinant $\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad - bc$
  $$\left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right| = a(ei - fh) - b(di - fg) + c(dh - eg)$$
- Fourier series (period L)
  $$f(x) = f(x+L) = c_0 + \sum_n c_n \cos\frac{2\pi nx}{L} + \sum_n d_n \sin\frac{2\pi nx}{L}$$
  Orthogonal condition
  $$\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\frac{2\pi nx}{L}\cos\frac{2\pi mx}{L}dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\frac{2\pi nx}{L}\cos\frac{2\pi mx}{L}dx = \frac{L}{2}\delta_{nm}$$
  $$\int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\frac{2\pi mx}{L}dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\frac{2\pi mx}{L}dx = \int_{-L/2}^{L/2} \cos\frac{2\pi nx}{L}\sin\frac{2\pi mx}{L}dx = 0$$
- Fourier transform
  $$f(k) = \frac{1}{2\pi}\int f(x)e^{-ikx}dx$$
  $$f(x) = \int f(k)e^{ikx}dk$$
  Orthogonal conditions:
  $$\int e^{i(k-k')x}dx = 2\pi\delta(k-k')f(x)$$
- Integrals:

$\int \sin mx\, dx = -\frac{1}{m}\cos mx + const.$ $\int \cos mx\, dx = \frac{1}{m}\sin mx + const.$

1. **Single harmonic oscillator (5 points each)**
A particle with mass m is confined in a harmonic potential $V(x) = \frac{1}{2}kx^2$ and the friction force is $-bx'(t)$, where $k > 0$ and $b > 0$. Answer the following questions. You do not need to show calculation.

A. If we release the particle at time $t = 0$ with $x(0) = x_0$ and $x'(0) = v_0$ and see underdamped oscillations. The motion will still be underdamped if we double the force constant and maintain the same initial condition.
Yes or No

B. Given a carefully tuned damping coefficient b, the oscillator can display a superposition of underdamped and overdamped motion.
Yes or No

C. In the presence of strong damping $b > 2\omega_0$, the so called over-damped regime, the particle will not display oscillatory motion even when it is driven by a periodic force $f(t) = \epsilon \cos \omega t$. The statement is
True or False

D. When the gravitational force is introduced to a vertically oriented oscillator with potential energy $V(z) = \frac{1}{2}kz^2 - mgz$, the frequency of the oscillator is
higher for larger g independent of g smaller for larger g

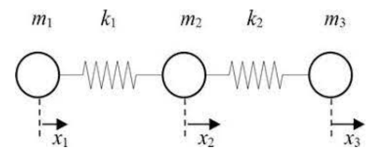
E. When an over-damped oscillator with natural frequency $\omega_0 \equiv \sqrt{k/m}$ is driven by a force $f(t) = f \cos \omega t$, the oscillator responds with the maximum amplitude when the driving frequency is
$\omega < \omega_0$ $\omega = \omega_0$ $\omega_0 < \omega \leq 2\omega_0$ $\omega > 2\omega_0$

F. You pick up a massive spring at one end (nothing is attached to the other end) and observe vertical oscillations of the spring with frequency ω_0 and negligible damping. Now you cut the spring in half and repeat the experiment, and the new oscillation frequency is
$\omega_0/2$ $\omega_0/\sqrt{2}$ also ω_0 $\sqrt{2}\omega_0$ $2\omega_0$

G. A pendulum in air is typically underdamped. Keeping the mass and air friction coefficient constant, can one possibly make the pendulum overdamped by changing the length of the string?
No way Yes, with a much longer string Yes, with a much shorter string

H. Which molecule has more normal modes?
O_2 CO_2 H_2O CH_3 All have infinite normal modes

2. **Coupled oscillators (10 points each)**
Three masses are connected by 2 springs with masses and force constants given in the figure on the right. They move in one dimension.



Introduce the displace vector $\vec{x}(t) \equiv \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$,

which follows the equation of motion (we ignore damping here)

$$\vec{x}''(t) = \hat{M}\vec{x}(t)$$

(a) Determine the matrix $\hat{M}$.

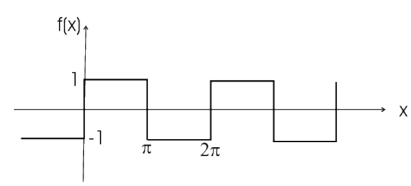
(b) Assume equal mass $m_1 = m_2 = m_3 \equiv m$ and equal force constant $k_1 = k_2 \equiv m\omega_0^2$. Determine the eigen-frequencies ω.

Hint: Use ansatz $\vec{x}(t) = \vec{A}e^{i\omega t}$, where $\vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$.

(c) Derive $\vec{A}$ and describe the motion of the normal modes.

3. **Fourier expansion and Fourier transform (10 points each)**
Let's work out the Fourier expansion and transform of the function

$$f(x) = \begin{cases} +1 & 0 < x < \pi \\ -1 & \pi < x < 2\pi, \end{cases}$$



with periodicity $L = 2\pi$, see Figure.
A. Start with the general expression, we have

$$f(x) = c_0 + \sum_{n=1} c_n \cos nx + \sum_{n=1} d_n \sin nx$$

Argue that $c_n = 0$ for all $n = 0, 1, 2 \ldots$

B. Derive $f(x) = \frac{4}{\pi}\sum_{m=1,3,5\ldots} \frac{\sin mx}{m}$.

C. Determine the Fourier transform of $f(x)$ using

$$f(k) = \frac{1}{2\pi}\int f(x)e^{-ikx}dx$$

Hint: you may use the result from B. to simplify the calculation.