Prove that $\int e^{ikx}dk = 2\pi\delta(x)$

$$\int_{-\infty}^{\infty} e^{ikx}dk = \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) e^{ikx}dk$$

$$= \int_0^\infty e^{-ikx} + e^{ikx}\, dk$$

We introduce a "regularization" $e^{-k\epsilon}$ so the integral converges @ ∞

$$= \lim_{\epsilon\to 0^+} \int_0^\infty e^{-ikx}e^{-k\epsilon} + e^{ikx}e^{-k\epsilon}\, dk$$

$$= \lim_{\epsilon\to 0^+} \int_0^\infty e^{(-ix-\epsilon)k} + e^{(ix-\epsilon)k}\, dk$$

$$= \lim_{\epsilon\to 0^+} \left. \frac{e^{(-ix-\epsilon)k}}{-ix-\epsilon} + \frac{e^{(ix-\epsilon)k}}{ix-\epsilon} \right|_0^\infty$$

$$= \lim_{\epsilon\to 0^+} \left(0 - \frac{1}{-ix-\epsilon}\right) + \left(0 - \frac{1}{ix-\epsilon}\right)$$

$$= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon+ix} + \frac{1}{\epsilon-ix} = \lim_{\epsilon\to 0^+} \frac{2\epsilon}{\epsilon^2+x^2}$$

Here we can see that $\lim_{\epsilon\to 0^+} \frac{2\epsilon}{\epsilon^2+x^2} = \begin{cases} 0 & \text{when } x\neq 0 \\ 2/\epsilon \to \infty & \text{when } x=0 \end{cases}$

So it behaves like a delta function $\delta(x)$, we just need to check

the normalization $\int \delta(x)dx = 1$. We have

$$\int \lim_{\epsilon\to 0^+} \frac{2\epsilon}{\epsilon^2+x^2}dx = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \frac{2\epsilon}{\epsilon^2+x^2}dx = \lim_{\epsilon\to 0^+} 2\tan^{-1}\frac{x}{\epsilon} = 2\pi$$

Thus $\int e^{ikx}dk = \lim_{\epsilon\to 0^+} \frac{2\epsilon}{\epsilon^2+x^2} = 2\pi\delta(x)$.